

# Jordan Pairs, $E_6$ and U-Duality in Five Dimensions

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## Abstract

By exploiting the *Jordan pair* structure of *U*-duality Lie algebras in  $D = 3$  and the relation to the *super-Ehlers* symmetry in  $D = 5$ , we elucidate the massless multiplet structure of the spectrum of a broad class of  $D = 5$  supergravity theories. Both *simple* and *semi-simple*, Euclidean rank-3 Jordan algebras are considered. Theories sharing the same bosonic sector but with different supersymmetrizations are also analyzed.

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## 1 Introduction

In recent past, exceptional Lie groups with their various real forms have been shown to play a major role in order to exploit several dynamical properties of supergravity theories in different dimensions.

Their relevance was highlighted by the seminal work of Cremmer and Julia [1], in which the existence of an exceptional electric-magnetic duality symmetry in  $\mathcal{N} = 8$ ,  $D = 4$  supergravity was established, based on the maximally non-compact (*split*) form  $E_{7(7)}$  of the exceptional group  $E_7$ .

Further advances were pioneered by Günaydin, Sierra and Townsend [2], which established the close relation between exceptional Lie groups occurring in  $D = 5$  supergravity theories and Jordan algebras. In particular, different real forms of  $E_6$  (namey, the split form  $E_{6(6)}$  for maximal  $\mathcal{N} = 8$  supergravity and the minimally non-compact form  $E_{6(-26)}$  for exceptional minimal Maxwell-Einstein  $\mathcal{N} = 2$  supergravity) made their appearance as *reduced structure* symmetries of the corresponding rank-3 Euclidean Jordan algebras. These latter are characterized by a cubic norm, which is directly related to *real special geometry* of the  $D = 5$  vector multiplets’ scalar manifold (see also *e.g.* [3], and Refs. therein) and to the Bekenstein-Hawking extremal  $D = 5$  black hole entropy, when the Jordan algebra elements are identified with the black hole charges (see *e.g.* [4, 5], and Refs. therein).

The  $U$ -duality<sup>1</sup> symmetry of maximal supergravity in  $D$  space-time dimensions is given by the

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<sup>1</sup>Here  $U$ -duality is referred to as the “continuous” symmetries of [1]. Their discrete versions are the  $U$ -duality non-

so-called *Cremmer-Julia sequence*  $E_{11-D(11-D)}$  [7, 8], which for  $D > 5$  yields classical groups. By arguments based on dimensional reduction, this sequence occurs in (at least) two sets of group embeddings for maximal supergravity in  $3 < D \leq 11$ :

$$E_{11-D(11-D)} \supset E_{10-D(10-D)} \times SO(1,1); \quad (1.1)$$

$$E_{8(8)} \supset E_{11-D(11-D)} \times SL(D-2, \mathbb{R}). \quad (1.2)$$

It is here worth remarking that, while (1.1) is maximal and symmetric, the same does not generally hold for (1.2).

The present investigation is devoted to the study of the embedding (1.2) in  $D = 5$ :

$$E_{8(8)} \supset E_{6(6)} \times SL(3, \mathbb{R}), \quad (1.3)$$

which, as recently pointed out in [9], is related to the so-called *Jordan pairs*. As also recently discussed in [10], for non-maximal supersymmetry, (1.3) is generalized as

$$G_N^3 \supset G_N^5 \times SL(3, \mathbb{R}), \quad (1.4)$$

where  $G_N^3$  and  $G_N^5$  respectively are the  $D = 3$  and  $D = 5$  *U*-duality groups of the theory with  $2N$  supersymmetries. For instance, in the minimal ( $N = 4$ , corresponding to  $\mathcal{N} = 2$  supercharges in  $D = 5$ ) exceptional supergravity [2], (1.4) specifies to a different non-compact, real form of (1.4), namely:

$$E_{8(-24)} \supset E_{6(-26)} \times SL(3, \mathbb{R}). \quad (1.5)$$

As recently analyzed in [10], the  $SL(3, \mathbb{R})$  appearing in (1.3) and (1.4) can physically be interpreted as the *Ehlers group* in  $D = 5$ . More generally, three decades ago, it was shown [11] that the  $D$ -dimensional *Ehlers group*  $SL(D-2, \mathbb{R})$  is a symmetry of  $D$ -dimensional Einstein gravity, provided that the theory is formulated in the light-cone gauge.

Interestingly enough, the supermultiplet structure of the underlying theory enjoys a natural explanation in terms of the *Jordan pair* embedding (1.4), if one considers the corresponding embedding of the maximal compact subgroups, which may largely differ depending on the relevant non-compact, real form. For example, in the maximal and minimal exceptional cases, the maximal compact level of (1.3) and of (1.5) respectively yields

$$SO(16) \supset Usp(8) \times SU(2)_J; \quad (1.6)$$

$$E_{7(-133)} \times SU(2) \supset F_{4(-52)} \times SU(2)_J, \quad (1.7)$$

where the  $SU(2)_J$  on the r.h.s. (maximal compact subgroup of the Ehlers  $SL(3, \mathbb{R})$ ) is the massless spin (helicity) group in  $D = 5$ .

The plan of the paper is as follows.

In Sec. 2, we introduce the  $q$ -parametrized sequence of exceptional Lie algebras, and its decomposition in terms of *Jordan pairs*, by starting from the treatment of the compact case recently given in [9], and pointing out the relation to *simple*, Euclidean rank-3 Jordan algebras. Suitable non-compact, real forms, relevant for application to the *super-Ehlers* symmetry [10] in  $D = 5$  supergravity, are considered.

In particular, Sec. 3 deals with *maximal* supergravity, related to  $\mathfrak{J}_3^{\oplus s}$  (Subsec. 3.1), and with *minimal* exceptional magical supergravity, related to  $\mathfrak{J}_3^{\oplus}$  [2]. In the latter case, considered in Subsec. 3.2, the existence of a  $D$ -independent hypersector is crucial to recover the massless multiplet structure *via* representation theory.

The interesting case of a pair of supergravity theory sharing the same bosonic sector, but with a different fermionic sector and thus with different supersymmetry properties, is considered in Sec. 4;

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perturbative string theory symmetries introduced by Hull and Townsend [6].

namely,  $\mathcal{N} = 6$  “pure” theory *versus* minimal ( $\mathcal{N} = 2$ ) matter-coupled Maxwell-Einstein quaternionic supergravity, both related to  $\mathfrak{J}_3^{\mathbb{H}}$  [2].

Sec. 5 lists the *Jordan pair* embeddings for all *simple*, Euclidean rank-3 Jordan algebras, also including the non-generic case of the  $D = 5$  uplift of the so-called  $T^3$  model ( $q = -2/3$ ).

*Semi-simple*, Euclidean rank-3 Jordan algebras are then analyzed in Sec. 6, focussing on the two infinite classes relevant for minimal and half-maximal supergravity in  $D = 5$  (the former class includes the  $D = 5$  uplift of the so-called *STU* model, which is separately analyzed in Subsec. 6.1.1).

Within the *semi-simple* framework, a pair of theories with the same bosonic sector but different supersymmetry features (namely minimal  $\mathfrak{J}_3^{2,6}$ -related theory *versus* half-maximal  $\mathfrak{J}_3^{6,2}$ -related theory, both matter coupled) is then analyzed in detail in Sec. 7.

Final observations and remarks are given in the concluding Sec. 8.

## 2 Jordan Pairs : the Simple Case

We start by briefly recalling that a *Jordan algebra*  $\mathfrak{J}$  [12, 13] is a vector space defined over a ground field  $\mathbb{F}$  equipped with a bilinear product  $\circ$  satisfying

$$\begin{aligned} X \circ Y &= Y \circ X; \\ X^2 \circ (X \circ Y) &= X \circ (X^2 \circ Y), \quad \forall X, Y \in \mathfrak{J}. \end{aligned} \tag{2.1}$$

The Jordan algebras relevant for the present investigation are *rank-3* Jordan algebras  $\mathfrak{J}_3$  over  $\mathbb{F} = \mathbb{R}$ , which come equipped with a cubic norm

$$\begin{aligned} N &: \mathfrak{J} \rightarrow \mathbb{R}, \\ N(\lambda X) &= \lambda^3 N(X), \quad \forall \lambda \in \mathbb{R}, X \in \mathfrak{J}. \end{aligned} \tag{2.2}$$

As an example, we anticipate that for both the rank-3 Jordan algebras  $\mathfrak{J}_3^{\mathbb{O}}$  and  $\mathfrak{J}_3^{\mathbb{O}_s}$  treated in Sec. 3, the relevant vector space is the representation space **27** pertaining to the fundamental irrep. of  $E_{6(-26)}$  resp.  $E_{6(6)}$ , and the cubic norm  $N$  is realized in terms of the completely symmetric invariant rank-3 tensor  $d_{IJK}$  in the **27** ( $I, J, K = 1, \dots, 27$ ):

$$(\mathbf{27} \times \mathbf{27} \times \mathbf{27})_s \ni \exists! 1 \equiv d_{IJK}; \tag{2.3}$$

$$N(X) \equiv d_{IJK} X^I X^J X^K. \tag{2.4}$$

There is a general prescription for constructing rank-3 Jordan algebras, due to Freudenthal, Springer and Tits [14, 15, 16], for which all the properties of the Jordan algebra are essentially determined by the cubic norm  $N$  (for a sketch of the construction see also [17]).

The  $q$ -parametrized sequence of  $U$ -duality (non-compact, real) Lie algebras  $\mathfrak{L}^q$  in  $D = 3$  (Lorentzian) space-time dimensions can be characterized as follows:

$$\mathfrak{L}^q = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{str}_0(\mathfrak{J}_3^q) \oplus \mathbf{3} \times \mathfrak{J}_3^q \oplus \mathbf{3}' \times \mathfrak{J}_3^{q'}, \tag{2.5}$$

where  $\mathfrak{J}_3^q$  is a rank-3 Euclidean *simple* Jordan algebra; for the cases  $q = 8, 4, 2, 1$ , the parameter  $q$  is defined as  $q \equiv \dim_{\mathbb{R}} \mathbb{A}$ , with  $\mathbb{A}$  denoting one of the four normed division algebras  $\mathbb{A} = \mathbb{O}, \mathbb{H}, \mathbb{C}, \mathbb{R}$  (from the famous “1,2,4,8” Hurwitz’s Theorem; see *e.g.* [16]), respectively.  $\mathfrak{J}_3^q$  fits into a  $(3q + 3)$ -dimensional irrep. of the *reduced structure* Lie algebra  $\mathfrak{str}_0(\mathfrak{J}_3^q)$ , which is nothing but the  $D = 5$   $U$ -duality Lie algebra. Also,

$$\mathfrak{L}^q = \mathfrak{qconf}(\mathfrak{J}_3^q) \tag{2.6}$$

is the *quasi-conformal* algebra of  $\mathfrak{J}_3^q$  [18, 19], *i.e.* the  $U$ -duality Lie algebra in  $D = 3$  (see *e.g.* [20, 21] for an introduction to the application of Jordan algebras and their symmetries in supergravity<sup>2</sup>, and lists of Refs.).

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<sup>2</sup>In these theories, the  $U$ -duality Lie algebra in  $D = 4$  is given by  $\mathfrak{conf}(\mathfrak{J}_3) = \mathfrak{aut}(\mathfrak{F}(\mathfrak{J}_3))$ , where  $\mathfrak{F}(\mathfrak{J}_3)$  denotes the *Freudenthal triple system* constructed over  $\mathfrak{J}_3$ .

(2.5) is a suitable non-compact, real version of the decomposition of the compact Lie algebras (subscript “ $c$ ” stands for *compact*) [9]

$$\mathfrak{L}_c^q = \mathfrak{su}(3) \oplus \mathfrak{str}_{0,c}(\mathfrak{J}_3^q) \oplus \mathbf{3} \times \mathfrak{J}_3^q \oplus \overline{\mathbf{3}} \times \overline{\mathfrak{J}_3^q}, \quad (2.7)$$

with the various cases given by the following Table 1: The sequence  $\mathfrak{L}_c^q$  is usually named “*exceptional*”

$q$	8	4	2	1	0	$-2/3$	$-1$
$\mathfrak{L}_c^q$	$\mathfrak{e}_{8(-248)}$	$\mathfrak{e}_{7(-133)}$	$\mathfrak{e}_{6(-78)}$	$\mathfrak{f}_{4(-52)}$	$\mathfrak{so}(8)$	$\mathfrak{g}_{2(-14)}$	$\mathfrak{su}(3)$
$\mathfrak{str}_{0,c}$	$\mathfrak{e}_{6(-78)}$	$\mathfrak{su}(6)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(3)$	$\mathfrak{su}(3)$	$\mathfrak{u}(1) \oplus \mathfrak{u}(1)$	—	—

sequence” (or “*exceptional series*”; see *e.g.* [22], and Refs. therein).

At Lie group level, the algebraic decompositions (2.5) and (2.7) are Cartan decompositions respectively pertaining to the following maximal non-symmetric embeddings:

$$QConf(\mathfrak{J}_3^q) \supset SL(3, \mathbb{R}) \times Str_0(\mathfrak{J}_3^q); \quad (2.8)$$

$$QConf_c(\mathfrak{J}_3^q) \supset SU(3) \times Str_{0,c}(\mathfrak{J}_3^q). \quad (2.9)$$

The non-semi-simple part of the r.h.s. of (2.5) and (2.7) is given by a pair of triplets of Jordan algebras, which is usually named “*Jordan pair*” (for a recent application in the compact case and a list of Refs., see *e.g.* [9]).

(Suitable real, non-compact forms of) all exceptional Lie algebras can be characterized as *quasi-conformal* algebras<sup>3</sup> of suitable Euclidean simple Jordan algebras of rank 3. Moreover, in the next Secs. we will consider the extension of *Jordan pairs* to *semi-simple* Euclidean Jordan algebras of rank 3 of relevance for supergravity theories (to which the case of  $\mathfrak{so}(8)$ ,  $q = 0$  belongs).

As recently analyzed in [10], the  $SL(3, \mathbb{R})$  appearing in (2.8) can physically be interpreted as the *Ehlers group* in  $D = 5$ . Three decades ago, it was shown [11] that the  $D$ -dimensional *Ehlers group*  $SL(D-2, \mathbb{R})$  is a symmetry of  $D$ -dimensional Einstein gravity, provided that the theory is formulated in the light-cone gauge. For any  $D \geq 4$ -dimensional Lorentzian space-time, this results enables to identify the graviton degrees of freedom with the Riemannian coset

$$\mathcal{M}_{grav} = \frac{SL(D-2, \mathbb{R})_{\text{Ehlers}}}{SO(D-2)_J}, \quad (2.10)$$

even if the action of the theory is not simply the sigma model action on this coset (with the exception of  $D = 3$ ). In  $D = 5$ , this statement reduces to the well known fact that the massless graviton described by the Einstein-Hilbert action with five degrees of freedom allows for an enhancement of the *massless* spin subgroup  $SO(3)_J \sim SU(2)_J$  of the Lorentz group in  $D = 5$  (Lorentzian) space-time dimensions the *non-compact* Ehlers group :

$$SU(2)_J \rightarrow SL(3, \mathbb{R})_{\text{Ehlers}}. \quad (2.11)$$

As studied *e.g.* in [23, 24, 25, 26, 27], in  $\mathcal{N}$ -extended supergravity theories in  $D$  dimensions, the Ehlers group enjoys an interesting interplay with the  $U$ -duality symmetry  $G_{\mathcal{N}}^D$ ; algebraically, it can be defined as the commutant of  $G_{\mathcal{N}}^D$  itself inside the  $D = 3$   $U$ -duality  $G_{\mathcal{N}}^3$ :

$$G_{\mathcal{N}}^3 \supset G_{\mathcal{N}}^D \times SL(D-2, \mathbb{R})_{\text{Ehler}}. \quad (2.12)$$

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<sup>3</sup>The case  $q = -1$  is trivial ( $\mathfrak{su}(3) = \mathfrak{su}(3)$ ), and it corresponds to “*pure*”  $\mathcal{N} = 2$ ,  $D = (3, 1)$  supergravity; therefore, it does not admit an uplift to five dimensions, and it will henceforth not be considered. Moreover,  $\mathfrak{su}(2)$  might be considered as  $q = -4/3$  element of the sequence in the second row of Table 1, as well. However, this is a limit case of the “*exceptional*” sequence reported in Table 1, not pertaining to Jordan pairs nor to supergravity in  $D = 3$  dimensions, and thus we will disregard it.

In [10], the direct product  $G_{\mathcal{N}}^D \times SL(D-2, \mathbb{R})$  was dubbed *super-Ehlers group*, and it was conjectured to be a manifest *off-shell* symmetry in the Hamiltonian light-cone formulation of the  $\mathcal{N}$ -extended supergravity theory.

In  $D = 5$ , the specification of (2.12) for all  $\mathcal{N} > 2$  theories, as well as for a broad class of  $\mathcal{N} = 2$  models, is given by (2.8) itself. This latter is a non-symmetric embedding, but it is however *maximal*; thus, no further “*enhancement*” of the super-Ehlers symmetry into some larger symmetry occurs<sup>4</sup>, as instead is the case in  $D = 10$  type IIB supergravity and other theories [10].

The present paper is devoted to the detailed analysis of suitable non-compact real form of *Jordan pairs*, and elucidation of their relevance for the algebraic definition of the super-Ehlers symmetry in  $D = 5$  supergravity theories, as well as for the determination of the multiplet structure of the massless spectrum.

### 3 $q = 8$

Let us consider the case  $q = 8$ . From (2.7) and Table 1, it corresponds to

$$\mathfrak{e}_{8(-248)} = \mathfrak{su}(3) \oplus \mathfrak{e}_{6(-78)} \oplus \mathbf{3} \times \mathbf{27} \oplus \bar{\mathbf{3}} \times \bar{\mathbf{27}}, \quad (3.1)$$

or, at compact group level:

$$E_{8(-248)} \supset SU(3) \times E_{6(-78)}; \quad (3.2)$$

$$\mathbf{248} = (\mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{78}) + (\mathbf{3}, \mathbf{27}) + (\bar{\mathbf{3}}, \bar{\mathbf{27}}), \quad (3.3)$$

where  $\mathbf{27}$  is the fundamental irrep. of  $E_{6(-78)}$ .

By confining ourselves to *Euclidean* rank-3 simple Jordan algebras, two possibility arise (recall (1.3) and (1.5)):

$$q = 8 : \begin{cases} \mathfrak{J}_3^{\oplus} : \begin{cases} \mathfrak{e}_{8(-24)} = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{e}_{6(-26)} \oplus \mathbf{3} \times \mathbf{27} \oplus \mathbf{3}' \times \mathbf{27}', \\ E_{8(-24)} \supset SL(3, \mathbb{R}) \times E_{6(-26)}; \end{cases} \\ \mathfrak{J}_3^{\oplus_s} : \begin{cases} \mathfrak{e}_{8(8)} = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{e}_{6(6)} \oplus \mathbf{3} \times \mathbf{27} \oplus \mathbf{3}' \times \mathbf{27}', \\ E_{8(8)} \supset SL(3, \mathbb{R}) \times E_{6(6)}, \end{cases} \end{cases} \quad (3.4)$$

where  $E_{8(-24)}$  and  $E_{8(8)}$  are the two only real, non-compact forms of  $E_8$ , namely the minimally non-compact and the maximally non-compact (*split*) one.

#### 3.1 $\mathfrak{J}_3^{\oplus_s}$

Let us start by considering the split case. This has an interpretation as *maximal* supergravity (32 supersymmetries) [2]; this is a “*pure*” theory, in which no matter coupling is allowed, and only the gravity multiplet exists.

The maximal compact subalgebra (*mcs*) of  $\mathfrak{qconf}(\mathfrak{J}_3^{\oplus_s}) = \mathfrak{e}_{8(8)}$  and  $\mathfrak{str}_0(\mathfrak{J}_3^{\oplus_s}) = \mathfrak{e}_{6(6)}$  respectively reads

$$mcs(\mathfrak{e}_{8(8)}) = \mathfrak{so}(16); \quad mcs(\mathfrak{e}_{6(6)}) = \mathfrak{usp}(8), \quad (3.5)$$

and the corresponding relevant maximal non-symmetric embedding is (recall (1.6))

$$\mathfrak{so}(16) = \mathfrak{su}(2) \oplus \mathfrak{usp}(8) \oplus \mathbf{3} \times \mathbf{27}; \quad (3.6)$$

$$SO(16) \supset SU(2) \times USp(8); \quad (3.7)$$

$$\mathbf{120} = (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{36}) + (\mathbf{3}, \mathbf{27}), \quad (3.8)$$

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<sup>4</sup>However, enhancement to infinite-dimensional Lie algebras, along the lines of [28], should occur.

where **27** is the rank-2 antisymmetric skew-traceless irrep. of  $USp(8)$ . Note that  $SO(16)$  and  $USp(8)$  are the  $\mathcal{R}$ -symmetry of  $\mathcal{N} = 16$ ,  $D = 3$  [29, 30] and of  $\mathcal{N} = 8$ ,  $D = 5$  [31] maximal supergravity, respectively.

On the other hand, the branchings corresponding to the maximal symmetric embeddings (3.5) read

$$E_{8(8)} \supset SO(16) : \mathbf{248} = \mathbf{120} + \mathbf{128}, \quad (3.9)$$

$$E_{6(6)} \supset USp(8) : \mathbf{78} = \mathbf{36} + \mathbf{42}. \quad (3.10)$$

In (3.9), **128** is one of the two chiral spinor irreps. of  $SO(16)$ , in which the generators of the rank-8 symmetric scalar coset  $\frac{E_{8(8)}}{SO(16)}$  of  $\mathcal{N} = 16$ ,  $D = 3$  maximal supergravity sit. In (3.10), **42** is the rank-4 antisymmetric skew-traceless self-real irrep. of  $USp(8)$ , in which the generators of the rank-6 symmetric scalar coset  $\frac{E_{6(6)}}{USp(8)}$  of  $\mathcal{N} = 8$ ,  $D = 5$  maximal supergravity sit.

Thus, under (3.6)-(3.7), it is worth considering also the following branchings:

$$SO(16) \supset SU(2) \times USp(8); \quad (3.11)$$

$$\mathbf{16} = (\mathbf{2}, \mathbf{8});$$

$$\mathbf{128} = (\mathbf{5}, \mathbf{1}) + (\mathbf{3}, \mathbf{27}) + (\mathbf{1}, \mathbf{42}); \quad (3.12)$$

$$\mathbf{128}' = (\mathbf{4}, \mathbf{8}) + (\mathbf{2}, \mathbf{48}), \quad (3.13)$$

where **48** is the rank-3 antisymmetric skew-traceless irrep. of  $USp(8)$ , and **128'** is the other chiral spinor irrep. of  $SO(16)$ , conjugate to **128**.

Some remarks are in order.

1. The branchings (3.13) and (3.12) suggests the identification of the  $SU(2)$  on the right-hand side of (3.7) (or (3.11)) as the *spin group* for massless particles (as understood throughout the present investigation) in  $D = 5$  space-time dimensions:

$$SU(2) \equiv SU(2)_J. \quad (3.14)$$

Indeed, decomposition (3.12) corresponds to the massless bosonic spectrum of  $\mathcal{N} = 8$ ,  $D = 5$  maximal supergravity (128 states): 1 spin-2 field (graviton), 27 spin-1 fields (graviphotons), and 42 spin-0 fields (real scalars). On the other hand, decomposition (3.13) yields the corresponding massless fermionic spectrum (128 states): 8 spin-3/2 fields (gravitinos) and 48 spin-1/2 fields (dilatinos). Thus, at the level of massless spectrum, the action of supersymmetry amounts to the following exchange of irreps.:

$$SO(16) : \mathbf{128}_B \longleftrightarrow \mathbf{128}'_F. \quad (3.15)$$

2. As denoted by the subscript “ $P$ ” in (3.16), the spin group  $SU(2)_J$  commuting with  $USp(8)$  inside  $SO(16)$  (recall (3.7) or (3.11)) is the Kostant “*principal*”  $SU(2)$  [32] maximally embedded into the  $SL(3, \mathbb{R})$  Ehlers group, which occurs in the embedding  $E_{8(8)} \supset SL(3, \mathbb{R}) \times E_{6(6)}$  pertaining to  $\mathfrak{J}_3^{\mathbb{O}_s}$  in (3.4):

$$SL(3, \mathbb{R}) \supset_P SU(2)_J : \mathbf{3} = \mathbf{3}, \mathbf{8} = \mathbf{3} + \mathbf{5}. \quad (3.16)$$

Due to the isomorphisms  $SU(2) \sim SO(3)$  and to the split nature of the non-compact, real form  $SL(3, \mathbb{R})$  of  $SU(3)$ , the maximal embedding (3.16) is *symmetric* (whereas generally the principal  $SU(2)$  embedding is non-symmetric). Therefore, consistent with its physical interpretation as Ehlers group in  $D = 5$  [11] (see also *e.g.* App. of [33]), the *split* form  $SL(3, \mathbb{R})$  of the Jordan-pair  $SU(3)$  maximally enhances the massless spin group  $SU(2)_J$  in  $D = 5$ , as given by the principal embedding (3.16).

3. As a consequence of the  $\mathfrak{J}_3^{\oplus_s}$ -related embedding in (3.4) and of the embedding (3.7) (or (3.11)), the following (maximal, non-symmetric) manifold embedding holds:

$$\frac{E_{8(8)}}{SO(16)} \supset \frac{E_{6(6)}}{USp(8)} \times \frac{SL(3, \mathbb{R})_{\text{Ehlers}}}{SU(2)_J}. \quad (3.17)$$

This has the trivial interpretation of embedding of the scalar manifold of ( $\mathcal{N} = 8$ ) maximal  $D = 5$  theory into the scalar manifold of the corresponding ( $\mathcal{N} = 16$ ) maximal theory in  $D = 3$ , obtained *e.g.* by two consecutive space-like Kaluza-Klein dimensional reductions. By recalling (2.10), the maximal symmetric rank-2 5-dimensional coset

$$\frac{SL(3, \mathbb{R})_{\text{Ehlers}}}{SU(2)_J} \sim \frac{SL(3, \mathbb{R})}{SO(3)} \quad (3.18)$$

in the r.h.s. of (3.17) is associated to the massless graviton degrees of freedom in  $D = 5$  Lorentzian space-time dimension. Indeed, it is nothing but the  $D = 5$  case of the coset (2.10).

4. Decompositions (3.12) and (3.13) of **128** and its conjugate **128'** under the embedding (3.11), which are consistent with the space-time spin-statistics, are not the usual ones, as reported *e.g.* in [34] and [35]. As investigated in [36] and [38] (see also [37] and, for a recent discussion, [10]), in the Lie algebra  $\mathfrak{so}(2n)$  ( $n \in \mathbb{N}$ ) there are pairs of subalgebras which are *inequivalent*, namely which are not mapped one into the other by the conjugation by an element of  $\mathfrak{so}(2n)$  itself. They are however *linearly equivalent*, *i.e.* in every representation of  $\mathfrak{so}(2n)$  they are mapped one into the other by a suitable implementation of the outer  $\mathfrak{so}(2n)$ -automorphism. Clearly, a (semi-)spinor irrep. of  $\mathfrak{so}(2n)$  branches differently into each of such two linearly-equivalent and inequivalent subalgebras. The cases relevant in the present investigation are obtained by setting  $n = 4$  and  $n = 3$  in the following maximal non-symmetric embedding pattern<sup>5</sup>

$$\begin{aligned} SO(4n) &\supset SU(2) \times USp(2n); \\ \mathbf{4n} &= (\mathbf{2}, \mathbf{2n}). \end{aligned} \quad (3.19)$$

From a Theorem due to Dynkin [36, 37], this embedding is nothing but a consequence of the *self-conjugacy* of the bi-fundamental  $(\mathbf{2}, \mathbf{2n})$  irrep. of  $SU(2) \times USp(2n)$ :

$$\begin{aligned} (\mathbf{2}, \mathbf{2n}) \times_s (\mathbf{2}, \mathbf{2n}) &= (\mathbf{2} \times_s \mathbf{2}, \mathbf{2n} \times_s \mathbf{2n}) + (\mathbf{2} \times_a \mathbf{2}, \mathbf{2n} \times_a \mathbf{2n}) \\ &= (\text{Adj}_{SU(2)}, \text{Adj}_{USp(2n)}) + (\mathbf{1}, \Lambda_0^2) + (\mathbf{1}, \mathbf{1}); \end{aligned} \quad (3.20)$$

$$\begin{aligned} (\mathbf{2}, \mathbf{2n}) \times_a (\mathbf{2}, \mathbf{2n}) &= (\mathbf{2} \times_s \mathbf{2}, \mathbf{2n} \times_a \mathbf{2n}) + (\mathbf{2} \times_a \mathbf{2}, \mathbf{2n} \times_s \mathbf{2n}) \\ &= (\text{Adj}_{SU(2)}, \mathbf{1}) + (\mathbf{1}, \text{Adj}_{USp(2n)}) + (\text{Adj}_{SU(2)}, \Lambda_0^2), \end{aligned} \quad (3.21)$$

where  $\Lambda_0^2$  is the rank-2 antisymmetric skew-traceless irrep. of  $USp(2n)$  (of total real dimension  $2n^2 - n - 1$ ). In general, two non-equivalent (but linearly equivalent)  $\mathfrak{usp}(2n)$  subalgebras of  $\mathfrak{so}(4n)$  exist (distinguished by a “+” or “−” subscript), under which the (semi-)spinor irreps. of  $\mathfrak{so}(4n)$  branch in different way. In particular, the case  $n = 4$  of (3.19) splits into a “standard” embedding (as *e.g.* reported in [34] and in [35]) pertains to, say,  $USp(8)_+$ , and it reads

$$SO(16) \supset SU(2) \times USp(8)_+; \quad (3.22)$$

$$\mathbf{128} = (\mathbf{4}, \mathbf{8}) + (\mathbf{2}, \mathbf{48}); \quad (3.23)$$

$$\mathbf{128}' = (\mathbf{5}, \mathbf{1}) + (\mathbf{3}, \mathbf{27}) + (\mathbf{1}, \mathbf{42}), \quad (3.24)$$

as well as into a “non-standard” embedding, pertaining to  $USp(8)_-$ , which is given by (3.11)-(3.13). As discussed above, this latter is relevant (for consistency of spin-statistics assignments

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<sup>5</sup>For the first application of such an embedding in supersymmetry, see *e.g.* [39].



in Lorentzian space-time) to  $J_3^{\mathbb{O}_s}$ , and thus to maximal supergravity in  $D = 5$ . It is immediate to realize that the role of the conjugate semi-spinor irreps. **128** and **128'** of  $SO(16)$  is interchanged in the “standard” and “non-standard” embeddings, or equivalently, when decomposed with respect to the maximal (singular) subalgebras  $USp(8)_+$  and  $USp(8)_-$ .

5. As recently analyzed in [10], to each (not necessarily maximal nor symmetric) embedding (2.12) one can associate a pseudo-Riemannian and a Riemannian compact coset, respectively:

$$M_{\mathcal{N}}^D \equiv \frac{G_{\mathcal{N}}^3}{G_{\mathcal{N}}^D \times SL(D-2, \mathbb{R})_{\text{Ehlers}}}; \quad (3.25)$$

$$\widehat{M}_{\mathcal{N}}^D \equiv \frac{mcs(G_{\mathcal{N}}^3)}{mcs(G_{\mathcal{N}}^D) \times SO(D-2)_J}, \quad (3.26)$$

where here “*mcs*” stands for maximal compact subgroup. In all theories of supergravity with symmetric scalar manifolds (as considered in [10]), the cosets  $M_{\mathcal{N}}^D$  (3.25) all have *vanishing character*, namely they have the same number of compact and non-compact generators, which in turn equals the real dimension of  $\widehat{M}_{\mathcal{N}}^D$  (3.26):

$$c(M_{\mathcal{N}}^D) = nc(M_{\mathcal{N}}^D) = \dim_{\mathbb{R}}(\widehat{M}_{\mathcal{N}}^D). \quad (3.27)$$

In [10], this property was related to Poincaré duality and to the symmetry of the cohomology of  $M_{\mathcal{N}}^D$  under the action of the Hodge involution. In  $\mathcal{N} = 8$ ,  $D = 5$  supergravity, (3.25)-(3.27) respectively specify to

$$M_{\mathcal{N}=8}^5 \equiv \frac{E_{8(8)}}{E_{6(6)} \times SL(3, \mathbb{R})_{\text{Ehlers}}}; \quad (3.28)$$

$$\widehat{M}_{\mathcal{N}=8}^5 \equiv \frac{SO(16)}{USp(8) \times SO(3)_J}; \quad (3.29)$$

$$c(M_{\mathcal{N}=8}^5) = nc(M_{\mathcal{N}=8}^5) = \dim_{\mathbb{R}}(\widehat{M}_{\mathcal{N}=8}^5) = 81, \quad (3.30)$$

and the result (3.30) can be simply explained by noticing that the Cartan decomposition pertaining to  $\widehat{M}_{\mathcal{N}=8}^5$  is given by (3.21) with  $n = 4$ , which thus yields that the generators of  $\widehat{M}_{\mathcal{N}=8}^5$  fit into the irrep.  $(\mathbf{Adj}_{SU(2)}, \mathbf{\Lambda}_0^2) = (\mathbf{3}, \mathbf{27})$  of  $SO(3)_J \times USp(8)$ , of total real dimension 81.

### 3.2 $\mathfrak{J}_3^{\mathbb{O}}$

The case pertaining to the rank-3 Euclidean Jordan algebra over the normed division algebra of octonions  $\mathbb{O}$  has an interpretation as *minimal* supergravity (8 supersymmetries), namely octonionic (also named exceptional) *magical* Maxwell-Einstein supergravity [2]. Coupling is allowed to two types of matter multiplets, namely vector and hyper multiplets.

An important difference with the case of  $\mathfrak{J}_3^{\mathbb{O}_s}$  treated above is the presence of an hypermultiplet sector, which is independent on the dimension  $D = 3, 4, 5, 6$  in which the theory is considered. The presence of such a  $D$ -independent hypersector enhances the gravity  $\mathcal{R}$ -symmetry of  $\mathcal{N} = 4$ ,  $D = 3$   $\mathfrak{J}_3^{\mathbb{O}}$ -related (*exceptional*) supergravity from the quaternionic  $SU(2)_H$  (related to the  $c$ -map of the  $D = 4$  vector multiplets’ scalar manifold) to  $SU(2)_H \times SU(2)'$ :

$$SU(2)_H \longrightarrow SU(2)_H \times SU(2)' \sim SO(4). \quad (3.31)$$

The maximal compact subalgebra (*mcs*) of  $\mathfrak{qconf}(\mathfrak{J}_3^{\mathbb{O}}) = \mathfrak{e}_{8(-24)}$  and  $\mathfrak{str}_0(\mathfrak{J}_3^{\mathbb{O}}) = \mathfrak{e}_{6(-26)}$  respectively reads

$$mcs(\mathfrak{e}_{8(-24)}) = \mathfrak{e}_{7(-133)} \oplus \mathfrak{su}(2)_H; \quad mcs(\mathfrak{e}_{6(-26)}) = \mathfrak{f}_{4(-52)}, \quad (3.32)$$

but the corresponding relevant maximal non-symmetric embedding must also include the  $\mathfrak{su}(2)'$  from the  $D$ -independent hypersector (recall (1.7)):

$$\begin{aligned}\mathfrak{e}_{7(-133)} \oplus \mathfrak{so}(4) &\sim \mathfrak{e}_{7(-133)} \oplus \mathfrak{su}(2)_H \oplus \mathfrak{su}(2)' \\ &= \mathfrak{f}_{4(-52)} \oplus \mathfrak{su}(2)_{\mathfrak{e}_7} \oplus \mathfrak{su}(2)_H \oplus \mathfrak{su}(2)' \\ &\oplus \mathbf{26} \times \mathbf{3} \times \mathbf{1} \times \mathbf{1};\end{aligned}\tag{3.33}$$

$$\begin{aligned}E_{7(-133)} \times SO(4) &\sim E_{7(-133)} \times SU(2)_H \times SU(2)' \\ &\supset F_{4(-52)} \times SU(2)_{E_7} \times SU(2)_H \times SU(2)';\end{aligned}\tag{3.34}$$

$$\begin{aligned}(\mathbf{133}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}) &= (\mathbf{52}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}) \\ &+ (\mathbf{26}, \mathbf{3}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3}),\end{aligned}\tag{3.35}$$

where  $\mathbf{26}$  is the fundamental irrep. of  $F_{4(-52)}$ . As mentioned,  $SU(2)_H \times SU(2)' \sim SO(4)$  (3.31) and  $SU(2)' \sim USp(2)$  are the  $\mathcal{R}$ -symmetry of  $\mathcal{N} = 4$ ,  $D = 3$  *exceptional* and of its uplift to  $D = 5^6$ , respectively. The group  $SU(2)_{E_7}$  is the one commuting with  $F_{4(-52)}$  in the maximal non-symmetric embedding

$$E_{7(-133)} \supset F_{4(-52)} \times SU(2)_{E_7},\tag{3.36}$$

determining (3.34)-(3.35).

Clearly, it holds that

$$\mathfrak{su}(2)' \cap \mathfrak{e}_{8(-24)} = \emptyset \Rightarrow \mathfrak{su}(2)' \cap \mathfrak{su}(2)_J = \emptyset; \quad \mathfrak{su}(2)' \cap \mathfrak{sl}(3, \mathbb{R}) = \emptyset;\tag{3.37}$$

$$\mathfrak{su}(2)_{\mathfrak{e}_7} \oplus \mathfrak{su}(2)_H \not\subset \mathfrak{sl}(3, \mathbb{R}).\tag{3.38}$$

As for the case of  $\mathfrak{J}_3^{\oplus s}$  treated above (recall (3.16)), the  $D = 5$  Ehlers Lie algebra  $\mathfrak{sl}(3, \mathbb{R})$  admits the massless spin algebra  $\mathfrak{su}(2)_J$  as maximal compact subalgebra. Due to the different multiplet structure, this latter is defined in a slightly more involved way with respect to the  $\mathfrak{J}_3^{\oplus s}$ -related maximally supersymmetric case treated above.

The branchings corresponding to the maximal symmetric embeddings (3.32) read

$$E_{8(-24)} \supset E_{7(-133)} \times SU(2)_H : \mathbf{248} = (\mathbf{133}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{56}, \mathbf{2});\tag{3.39}$$

$$E_{6(-26)} \supset F_{4(-52)} : \mathbf{78} = \mathbf{52} + \mathbf{26},\tag{3.40}$$

where  $(\mathbf{56}, \mathbf{2})$  is the bi-fundamental irrep. of  $E_{7(-133)} \times SU(2)_H$ , in which the generators of the rank-4 symmetric quaternionic scalar manifold  $\frac{E_{8(-24)}}{E_{7(-133)} \times SU(2)_H}$  of  $\mathcal{N} = 4$ ,  $D = 3$  exceptional supergravity sit. In (3.40), the generators of the rank-2 symmetric real special scalar manifold  $\frac{E_{6(-26)}}{F_{4(-52)}}$  of  $\mathcal{N} = 2$ ,  $D = 5$  exceptional supergravity sit in the fundamental irrep.  $\mathbf{26}$  of  $F_{4(-52)}$ . Thus, under (3.33)-(3.34), it is worth considering also the following branching:

$$\begin{aligned}E_{7(-133)} \times SU(2)_H \times SU(2)' &\supset F_{4(-52)} \times SU(2)_{E_7} \times SU(2)_H \times SU(2)'; \\ (\mathbf{56}, \mathbf{2}, \mathbf{1}) &= (\mathbf{1}, \mathbf{4}, \mathbf{2}, \mathbf{1}) + (\mathbf{26}, \mathbf{2}, \mathbf{2}, \mathbf{1}).\end{aligned}\tag{3.41}$$

Some remarks are in order.

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<sup>6</sup>This is a *unified*  $\mathcal{N} = 2$  theory, namely all vectors sit in an *irreducible* representation of the  $D = 5$   $U$ -duality group (in this case,  $\mathbf{27}$  of  $E_{6(-26)}$ ; see *e.g.* [40]).

1. In this theory, the massless spin group  $SU(2)_J$  in  $D = 5$  can be identified with the *diagonal*  $SU(2)$  maximally and symmetrically embedded into  $SU(2)_{E_7} \times SU(2)_H$ :

$$SU(2)_J \subset_d SU(2)_{E_7} \times SU(2)_H, \quad (3.42)$$

such that (3.41) can be completed to the following chain:

$$\begin{aligned} E_{7(-133)} \times SU(2)_H \times SU(2)' &\supset F_{4(-52)} \times SU(2)_{E_7} \times SU(2)_H \times SU(2)' \\ &\supset F_{4(-52)} \times SU(2)_J \times SU(2)'; \end{aligned} \quad (3.43)$$

$$\begin{aligned} (\mathbf{56}, \mathbf{2}, \mathbf{1}) &= (\mathbf{1}, \mathbf{4}, \mathbf{2}, \mathbf{1}) + (\mathbf{26}, \mathbf{2}, \mathbf{2}, \mathbf{1}) \\ &= (\mathbf{1}, \mathbf{5}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{26}, \mathbf{3}, \mathbf{1}) + (\mathbf{26}, \mathbf{1}, \mathbf{1}). \end{aligned} \quad (3.44)$$

Indeed, the decomposition (3.43) corresponds to the massless bosonic spectrum of  $\mathcal{N} = 2$ ,  $D = 5$  exceptional supergravity (112 states<sup>7</sup>): 1 graviton and 1 graviphoton from the gravity multiplet, and 26 vectors and 26 scalars from the 26 vector multiplets. At the level of massless spectrum, the action of supersymmetry amounts to the following exchange of irreps.:

$$E_{7(-133)} \times SU(2)_H \times SU(2)' : (\mathbf{56}, \mathbf{2}, \mathbf{1})_B \longleftrightarrow (\mathbf{56}, \mathbf{1}, \mathbf{2})_F. \quad (3.45)$$

Indeed, under (3.43),  $(\mathbf{56}, \mathbf{1}, \mathbf{2})$  decomposes as follows:

$$(\mathbf{56}, \mathbf{1}, \mathbf{2}) = (\mathbf{1}, \mathbf{4}, \mathbf{1}, \mathbf{2}) + (\mathbf{26}, \mathbf{2}, \mathbf{1}, \mathbf{2}) = (\mathbf{1}, \mathbf{4}, \mathbf{2}) + (\mathbf{26}, \mathbf{2}, \mathbf{2}), \quad (3.46)$$

thus reproducing the massless fermionic spectrum of  $\mathcal{N} = 2$ ,  $D = 5$  exceptional supergravity (112 states): 1  $SU(2)'$ -doublet of gravitinos, and 26  $SU(2)'$ -doublets of gauginos from the 26 vector multiplets. Note that, consistently, bosons are  $\mathcal{R}$ -symmetry  $SU(2)'$ -singlets, whereas fermions fit into  $SU(2)'$ -doublets. Note how the change of the kind of octonions ( $J_3^{\text{os}}$  versus  $J_3^{\text{O}}$ ) on which the bosonic theory is constructed affects its supersymmetrization as well as the relevant irreps. and the number of resulting massless states<sup>8</sup>: in (3.45), the chiral spinor irreps.  $\mathbf{128}$  and  $\mathbf{128}'$  of the maximal Clifford algebra  $SO(16)$  of (3.15) are not replaced by the tri-fundamental  $(\mathbf{56}, \mathbf{2}, \mathbf{2})$  of  $E_{7(-133)} \times SU(2)_H \times SU(2)'$ , but rather by the bi-fundamental  $(\mathbf{56}, \mathbf{2})$  of  $E_{7(-133)} \times SU(2)_H$  (for bosons) and by the bi-fundamental  $(\mathbf{56}, \mathbf{2})$  of  $E_{7(-133)} \times SU(2)'$  (for fermions).

2. As for the case of  $\mathfrak{J}_3^{\text{O}}$  treated above,  $SU(2)_J$ , which commutes with  $F_{4(-52)} \times SU(2)'$  inside  $E_{7(-133)} \times SU(2)_H \times SU(2)'$  (recall (3.43)), is the Kostant “*principal*”  $SU(2)$  (3.16) maximally embedded into the Ehlers group  $SL(3, \mathbb{R})$  group. Therefore, as for  $\mathfrak{J}_3^{\text{O}}$ , the split form  $SL(3, \mathbb{R})$  of the *Jordan-pair*  $SU(3)$  maximally enhances the  $D = 5$  massless spin group:

$$SL(3, \mathbb{R})_{\text{Ehlers}} \cap [SU(2)_{E_7} \times SU(2)_H] = SU(2)_J. \quad (3.47)$$

3. As a consequence of the  $\mathfrak{J}_3^{\text{O}}$ -related embedding in (3.4) and of the embedding (3.43), the following (non-maximal, non-symmetric) manifold embedding holds:

$$\frac{E_{8(-24)}}{E_{7(-133)} \times SU(2)_H} \supset \frac{E_{6(-26)}}{F_{4(-52)}} \times \frac{SL(3, \mathbb{R})_{\text{Ehlers}}}{SU(2)_J}. \quad (3.48)$$

This has the trivial interpretation of embedding of the scalar manifold of  $\mathcal{N} = 2$ ,  $D = 5$  theory into the scalar manifold of the theory dimensionally reduced to  $D = 3$  dimensions.

<sup>7</sup>In absence of  $D$ -independent hypermultiplets (as assumed throughout this paper for the theories with 8 local supersymmetries).

<sup>8</sup>In general, the number of massless bosonic states of the theory (in  $D = 3$  as well as in any dimension) is given by the (real) dimension of the irrep. of the  $mcs(G_{\mathcal{N}}^3)$  occurring in the Cartan decomposition of the  $D = 3$  scalar manifold. In supersymmetric theories, the numbers of bosonic and fermionic states coincide.

4. As resulting from the above treatment, the main difference between the  $\mathfrak{J}_3^{\oplus s}$  and  $\mathfrak{J}_3^{\oplus}$  cases resides in the  $D$ -independent hypersector. In the former case, pertaining to maximal supergravity, such a sector is forbidden by supersymmetry. In the latter case, pertaining to minimal supergravity, such a sector must be present for physical consistency; as mentioned above, this hypersector is insensitive to dimensional reductions, and it is thus independent on the number  $D = 3, 4, 5, 6$  of space-time dimensions in which the theory with 8 supersymmetries is defined<sup>9</sup>. In Lorentzian space-time signatures (which we consider throughout this paper), it introduces a  $D$ -independent  $SU(2)' \sim USp(2)$   $\mathcal{R}$ -symmetry, which enhances to  $SO(4) \sim SU(2)_H \times SU(2)'$  in  $D = 3$ . Note that  $SU(2)'$  is present also in absence of  $D$ -independent hypermultiplets (as we assume throughout this paper), in which case it is promoted to a *global* symmetry of the theory [42]. Moreover, in order to analyze the massless spectrum of the theory, such a  $D$ -independent hypersector does not need to be specified. By confining ourselves *e.g.* to symmetric hypermultiplets' scalar manifolds, they read

$$\frac{\mathcal{G}_3}{\mathcal{H}_3 \times SU(2)'}, \quad (3.49)$$

where  $\mathcal{H}_3 \times SU(2)'$  is the *mcs* of the  $D$ -independent hypersector global symmetry  $\mathcal{G}_3$ . The coset (3.49) is not necessarily the  $c$ -map [43] of the  $D = 4$  special Kähler vector multiplets' scalar manifold, as instead is the quaternionic manifold obtained as space-like KK reduction from the latter manifold (this parametrizes the scalar degrees of freedom of the genuinely  $D = 3$  hypersector). In the  $\mathfrak{J}_3^{\oplus}$ -related exceptional theory under consideration, the  $D = 4$  vector multiplets' scalar manifold is symmetric, and so is its  $c$ -map:

$$\frac{E_{7(-25)}}{E_{6(-78)} \times U(1)}_{D=4} \xrightarrow{c} \frac{E_{8(-24)}}{E_{7(-133)} \times SU(2)_H}_{D=3}. \quad (3.50)$$

For completeness, we recall that the  $\mathfrak{J}_3^{\oplus s}$  (maximally supersymmetric) analogue of (3.50) reads

$$\frac{E_{7(7)}}{SU(8)}_{D=4} \xrightarrow{c_m} \frac{E_{8(8)}}{SO(16)}_{D=3}, \quad (3.51)$$

where  $c_m$  is the “*maximal*” analogue of  $c$ -map, and  $\frac{E_{7(7)}}{SU(8)}$  is the rank-7 scalar symmetric coset of  $\mathcal{N} = 8$ ,  $D = 4$  maximal supergravity.

5. In  $\mathcal{N} = 2$ ,  $D = 5$  *exceptional* supergravity, (3.25)-(3.27) respectively specify to

$$M_{\mathcal{N}=2, J_3^{\oplus}}^5 \equiv \frac{E_{8(-24)}}{E_{6(-26)} \times SL(3, \mathbb{R})_{\text{Ehlers}}}; \quad (3.52)$$

$$\widehat{M}_{\mathcal{N}=2, J_3^{\oplus}}^5 \equiv \frac{E_{7(-133)} \times SU(2)_H}{USp(8) \times SU(2)_J}; \quad (3.53)$$

$$c\left(M_{\mathcal{N}=2, J_3^{\oplus}}^5\right) = nc\left(M_{\mathcal{N}=2, J_3^{\oplus}}^5\right) = \dim_{\mathbb{R}}\left(\widehat{M}_{\mathcal{N}=2, J_3^{\oplus}}^5\right) = 81. \quad (3.54)$$

Disregarding  $SU(2)'$ , the result (3.54) can be explained by noticing that the Cartan decomposition pertaining to  $\widehat{M}_{\mathcal{N}=2, J_3^{\oplus}}^5$  is given by further branching (3.35) with respect to (3.42):

$$E_{7(-133)} \times SU(2)_H \supset F_{4(-52)} \times SU(2)_{E_7} \times SU(2)_H \supset_d F_{4(-52)} \times SU(2)_J \quad (3.55)$$

$$\begin{aligned} & (\mathbf{133}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) \\ &= (\mathbf{52}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{26}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}), \\ &= (\mathbf{52}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{26}, \mathbf{3}) + (\mathbf{1}, \mathbf{3}), \end{aligned} \quad (3.56)$$

<sup>9</sup>We recall that the geometry of the hyperscalars in supergravity is quaternionic (and thus Einstein); see *e.g.* [41].

which thus yields that the generators of  $\widehat{M}_{\mathcal{N}=2, J_3^\oplus}^5$  fit into the sum of irreps.  $(\mathbf{26}, \mathbf{3}) + (\mathbf{1}, \mathbf{3})$  of  $F_{4(-52)} \times SU(2)_J$ , of total real dimension 81. Note that this is the same dimension obtained in the case of  $J_3^{\oplus s}$ , but with a different covariant decomposition.; in particular, the presence of the  $F_{4(-52)}$ -singlet  $(\mathbf{1}, \mathbf{3})$  (*graviphoton*) is related to the fact that the theory has 8 local supersymmetries.

## 4 $q = 4$ , $\mathfrak{J}_3^{\mathbb{H}}$ “Twin” Theories

As evident from the treatment above, the presence or absence of a  $D$ -independent hypersector is implied by the physical (supergravity) interpretation of the model under consideration. From a group theoretical perspective, of course one could have added an extra  $SU(2)'$  also in the treatment of  $\mathfrak{J}_3^{\oplus s}$ , or disregarded the  $D$ -independent hypersector in the treatment of  $\mathfrak{J}_3^\oplus$ . However, in both cases one would have failed to reproduce the massless spectrum of the corresponding supergravity theory.

In some special cases, dubbed “(bosonic) twin” theories, the  $D$ -independent hypersector can or cannot be considered, and in both instances the resulting supergravity theory (of course with different number of local supersymmetries) is physically meaningful. Indeed, “twin” theories share the very same bosonic sector, which is however supersymmetrized in (*at least*) two different ways [add Refs. on twin-theories].

A nice example of “twin” theories is provided by the  $q = 4$  case of  $\mathfrak{J}_3^{\mathbb{H}}$  [2] (see also *e.g.* [44, 45, 46, 47]) which we will now analyze.

From (2.7) and Table 1,  $q = 4$  corresponds to

$$\mathfrak{e}_{7(-133)} = \mathfrak{su}(3) \oplus \mathfrak{su}(6) \oplus \mathbf{3} \times \overline{\mathbf{15}} \oplus \overline{\mathbf{3}} \times \mathbf{15}, \quad (4.1)$$

or, at compact group level:

$$E_{7(-133)} \supset SU(3) \times SU(6); \quad (4.2)$$

$$\mathbf{133} = (\mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{35}) + (\mathbf{3}, \overline{\mathbf{15}}) + (\overline{\mathbf{3}}, \mathbf{15}), \quad (4.3)$$

where  $\mathbf{15}$  is the rank-2 antisymmetric irrep. of  $SU(6)$ .

By confining ourselves to *Euclidean* rank-3 simple Jordan algebras, two possibility arise:

$$q = 4 : \begin{cases} \mathfrak{J}_3^{\mathbb{H}} : \begin{cases} \mathfrak{e}_{7(-5)} = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{su}^*(6) \oplus \mathbf{3} \times \mathbf{15}' \oplus \mathbf{3}' \times \mathbf{15}, \\ E_{7(-5)} \supset SL(3, \mathbb{R}) \times SU^*(6); \end{cases} \\ \mathfrak{J}_3^{\mathbb{H}_s} : \begin{cases} \mathfrak{e}_{7(7)} = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(6, \mathbb{R}) \oplus \mathbf{3} \times \mathbf{15}' \oplus \mathbf{3}' \times \mathbf{15}, \\ E_{7(7)} \supset SL(3, \mathbb{R}) \times SL(3, \mathbb{R}). \end{cases} \end{cases} \quad (4.4)$$

As mentioned, we will here consider the case  $\mathfrak{J}_3^{\mathbb{H}}$ , relevant for “(bosonic) twin” theories.

### 4.1 24 Supersymmetries

Let us start by considering the physical interpretation of the  $\mathfrak{J}_3^{\mathbb{H}}$ -related model as theory with 24 local supersymmetries; namely, in the framework under consideration,  $\mathcal{N} = 6$ ,  $D = 5$  supergravity and its dimensional reduction ( $\mathcal{N} = 12$ ) to  $D = 3$  dimensions. They are “pure” theories : no matter coupling is allowed.

The *mcs* of  $\mathfrak{qconf}(\mathfrak{J}_3^{\mathbb{H}}) = \mathfrak{e}_{7(-5)}$  and  $\mathfrak{str}_0(\mathfrak{J}_3^{\mathbb{H}}) = \mathfrak{su}^*(6)$  respectively reads

$$mcs(\mathfrak{e}_{7(-5)}) = \mathfrak{so}(12) \oplus \mathfrak{su}(2)_{(H)}; \quad mcs(\mathfrak{su}^*(6)) = \mathfrak{usp}(6), \quad (4.5)$$

and the corresponding relevant maximal non-symmetric embedding is

$$\mathfrak{so}(12) \oplus \mathfrak{su}(2)_{(H)} = \mathfrak{usp}(6) \oplus \mathfrak{su}(2)_{\mathfrak{so}(12)} \oplus \mathbf{3} \times \mathbf{14} \oplus \mathfrak{su}(2)_{(H)}; \quad (4.6)$$

$$SO(12) \times SU(2)_{(H)} \supset USp(6) \times SU(2)_{SO(12)} \times SU(2)_{(H)}; \quad (4.7)$$

$$(\mathbf{66}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) = (\mathbf{21}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{14}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}), \quad (4.8)$$

where  $\mathbf{14}$  is the rank-2 antisymmetric skew-traceless irrep. of  $USp(6)$ . Note that  $SO(12) \times SU(2)_{(H)}$  and  $USp(6)$  are the  $\mathcal{R}$ -symmetry of  $\mathcal{N} = 12$ ,  $D = 3$  and of  $\mathcal{N} = 6$ ,  $D = 5$  “pure” supergravity, respectively. Here, the subscript “ $(H)$ ” denotes the fact that  $SU(2)_{(H)}$  actually is the quaternionic  $SU(2)$  in the physical interpretation pertaining to 8 local supersymmetries (see below). For later convenience, by  $SU(2)_{SO(12)}$  we denote the group commuting with  $USp(6)$  in the maximal non-symmetric embedding

$$SO(12) \supset USp(6) \times SU(2)_{SO(12)}, \quad (4.9)$$

determining (4.7).

On the other hand, the branchings corresponding to the maximal symmetric embeddings (4.5) read

$$E_{7(-5)} \supset SO(12) \times SU(2)_{(H)} : \mathbf{133} = (\mathbf{66}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{32}', \mathbf{2}), \quad (4.10)$$

$$SU^*(6) \supset USp(6) : \mathbf{35} = \mathbf{21} + \mathbf{14}, \quad (4.11)$$

where  $\mathbf{32}'$  is one of the two chiral spinor irreps. of  $SO(12)$ . The generators of the rank-4 quaternionic Kähler symmetric scalar manifold  $\frac{E_{7(-5)}}{SO(12) \times SU(2)_{(H)}}$  of  $\mathcal{N} = 12$ ,  $D = 3$  supergravity sit in the  $(\mathbf{32}', \mathbf{2})$ . On the other hand, the generators of the rank-2 real special symmetric scalar manifold  $\frac{SU^*(6)}{USp(6)}$  of  $\mathcal{N} = 6$ ,  $D = 5$  supergravity sit in the  $\mathbf{14}$  of  $USp(6)$ . Thus, under (4.6)-(4.7), it is worth considering also the following branchings:

$$SO(12) \times SU(2)_{(H)} \supset USp(6) \times SU(2)_{SO(12)} \times SU(2)_{(H)}; \quad (4.12)$$

$$(\mathbf{12}, \mathbf{1}) = (\mathbf{6}, \mathbf{2}, \mathbf{1});$$

$$(\mathbf{32}, \mathbf{2}) = (\mathbf{14}', \mathbf{1}, \mathbf{2}) + (\mathbf{6}, \mathbf{3}, \mathbf{2});$$

$$(\mathbf{32}', \mathbf{2}) = (\mathbf{14}, \mathbf{2}, \mathbf{2}) + (\mathbf{1}, \mathbf{4}, \mathbf{2});$$

where  $\mathbf{14}'$  is the rank-3 antisymmetric skew-traceless irrep. of  $USp(6)$ , and  $\mathbf{32}$  is the other chiral spinor irrep. of  $SO(12)$ , conjugate to  $\mathbf{32}'$ .

Some remarks are in order.

1. Branching (4.6) (or (4.12)) is consistent with the identification of the massless  $D = 5$  spin group with the diagonal  $SU(2)$  embedded into  $SU(2)_{SO(12)} \times SU(2)_{(H)}$ :

$$SU(2)_J \subset_d SU(2)_{SO(12)} \times SU(2)_{(H)}. \quad (4.13)$$

Thus, (4.6) (or (4.12)) can be completed to the following chain:

$$SO(12) \times SU(2)_{(H)} \supset USp(6) \times SU(2)_{SO(12)} \times SU(2)_{(H)} \supset USp(6) \times SU(2)_J; \quad (4.14)$$

$$\begin{aligned} (\mathbf{66}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) &= (\mathbf{21}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{14}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}) \\ &= (\mathbf{21}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{14} + \mathbf{1}, \mathbf{3}); \end{aligned} \quad (4.15)$$

$$(\mathbf{32}, \mathbf{2}) = (\mathbf{14}', \mathbf{1}, \mathbf{2}) + (\mathbf{6}, \mathbf{3}, \mathbf{2}) = (\mathbf{14}', \mathbf{2}) + (\mathbf{6}, \mathbf{4}) + (\mathbf{6}, \mathbf{2}); \quad (4.16)$$

$$(\mathbf{32}', \mathbf{2}) = (\mathbf{14}, \mathbf{2}, \mathbf{2}) + (\mathbf{1}, \mathbf{4}, \mathbf{2}) = (\mathbf{14}, \mathbf{3}) + (\mathbf{14}, \mathbf{1}) + (\mathbf{1}, \mathbf{5}) + (\mathbf{1}, \mathbf{3}). \quad (4.17)$$

Note that the  $(3q+3)_{q=4} = 15$ -dimensional rep. of  $USp(6)$  is reducible as  $\mathbf{14} + \mathbf{1}$ ; as recently discussed *e.g.* in [10], this is a peculiarity of the  $\mathcal{N} = 6$  theory, and allows for a different (“twin”)

supersymmetrization with only 8 local supersymmetries (see below). The decomposition (4.16) corresponds to the massless fermionic spectrum of  $\mathcal{N} = 6$ ,  $D = 5$  supergravity (64 states): 14 + 6 spin 1/2 fermions, and 6 gravitinos. On the other hand, the decomposition (4.17) corresponds to the massless bosonic spectrum (64 states): 1 graviton, 14 + 1 graviphotons, and 14 scalar fields. Thus, at the level of massless spectrum, the action of supersymmetry amounts to the following exchange of irreps.:

$$SO(12) \times SU(2)_{(H)} : (\mathbf{32}, \mathbf{2})_F \longleftrightarrow (\mathbf{32}', \mathbf{2})_B. \quad (4.18)$$

2. As for the cases of  $\mathfrak{J}_3^{\oplus}$  and  $\mathfrak{J}_3^{\oplus_s}$  treated above, and as holding true in general,  $SU(2)_J$ , which commutes with  $USp(6)$  inside  $SO(12) \times SU(2)_{(H)}$  (recall (4.14), is the Kostant “principal”  $SU(2)$  (3.16) into the  $D = 5$  Ehlers  $SL(3, \mathbb{R})$ :

$$SL(3, \mathbb{R})_{\text{Ehlers}} \cap \left[ SU(2)_{SO(12)} \times SU(2)_{(H)} \right] = SU(2)_J. \quad (4.19)$$

3. As a consequence of the  $\mathfrak{J}_3^{\mathbb{H}}$ -related embedding in (4.4) and of the embedding (4.6), the following (non-maximal, non-symmetric) manifold embedding holds:

$$\frac{E_{7(-5)}}{SO(12) \times SU(2)_{(H)}} \supset \frac{SU^*(6)}{USp(6)} \times \frac{SL(3, \mathbb{R})_{\text{Ehlers}}}{SU(2)_J}. \quad (4.20)$$

As above, this has the trivial interpretation of embedding of the scalar manifold of  $\mathcal{N} = 6$ ,  $D = 5$  theory into the scalar manifold of the corresponding theory reduced to  $D = 3$ .

4. Decompositions (4.16) and (4.17) of  $(\mathbf{32}, \mathbf{2})$  and its conjugate  $(\mathbf{32}', \mathbf{2})$  under the first embedding of (4.14), which are consistent with the space-time spin-statistics, are not the usual ones, as reported *e.g.* in [34] and [35]. In fact, the first embedding of (4.14) is nothing but the case  $n = 3$  of the embedding pattern discussed at point 4 of Subsec. 3.1. In particular, the case  $n = 3$  of (3.19) splits into a “standard” embedding (as *e.g.* reported in [34] and in [35]) pertains to, say,  $USp(6)_+$ , and it reads

$$SO(12) \supset SU(2)_{SO(12)} \times USp(6)_+; \quad (4.21)$$

$$\mathbf{32} = (\mathbf{2}, \mathbf{14}) + (\mathbf{4}, \mathbf{1}); \quad (4.22)$$

$$\mathbf{32}' = (\mathbf{1}, \mathbf{14}') + (\mathbf{3}, \mathbf{6}), \quad (4.23)$$

as well as into a “non-standard” embedding, pertaining to  $USp(6)_-$ , which is indeed given by the first step of (4.14) and (4.16)-(4.17). It is immediate to realize that the role of the conjugate semi-spinor irreps.  $\mathbf{32}$  and  $\mathbf{32}'$  of  $SO(12)$  is interchanged in the “standard” and “non-standard” embeddings, or equivalently, when decomposed with respect to the maximal (singular) subalgebras  $USp(6)_+$  and  $USp(6)_-$ .

5. In  $\mathcal{N} = 6$ ,  $D = 5$  supergravity, (3.25)-(3.27) respectively specify to

$$M_{\mathcal{N}=6}^5 \equiv \frac{E_{7(-5)}}{SU^*(6) \times SL(3, \mathbb{R})_{\text{Ehlers}}}; \quad (4.24)$$

$$\widehat{M}_{\mathcal{N}=6}^5 \equiv \frac{SO(12) \times SU(2)_H}{USp(6) \times SU(2)_J}; \quad (4.25)$$

$$c(M_{\mathcal{N}=6}^5) = nc(M_{\mathcal{N}=6}^5) = \dim_{\mathbb{R}}(\widehat{M}_{\mathcal{N}=6}^5) = 45. \quad (4.26)$$

The result (4.26) has been explained in [10] in terms of the Cartan decomposition of  $\widehat{M}_{\mathcal{N}=6}^5$ .

## 4.2 8 Supersymmetries

Let us proceed to considering the physical interpretation of the  $\mathfrak{J}_3^{\mathbb{H}}$ -related bosonic model as bosonic sector of a minimal supergravity theory (8 local supersymmetries); in the framework under consideration, involving *Jordan pairs*, this corresponds to  $\mathcal{N} = 2$ ,  $D = 5$  quaternionic *magical* Maxwell-Einstein supergravity<sup>10</sup> [2], and its dimensional reduction to  $D = 3$ . Matter coupling is allowed through two types of multiplets, namely vector and hyper multiplets.

A crucial difference with the case pertaining to 24 supersymmetries treated in previous Subsection is the presence of an hypermultiplet sector which is independent on the dimension  $D = 3, 4, 5, 6$  in which the quarter-minimal theory is considered. Such a  $D$ -independent hypersector enhances the  $\mathcal{R}$ -symmetry of  $\mathcal{N} = 4$ ,  $D = 3$  magical quaternionic supergravity from the quaternionic  $SU(2)_H$  (related to the  $c$ -map of the  $D = 4$  vector multiplets' scalar manifold) to  $SU(2)_H \times SU(2)'$ , as given by (3.31).

Therefore, the corresponding relevant maximal non-symmetric embedding must include the  $\mathfrak{su}(2)'$  algebra from the  $D$ -independent hypersector, also when hypermultiplets are actually absent (in this case,  $\mathfrak{su}(2)'$  is a global symmetry):

$$\begin{aligned} \mathfrak{so}(12) \oplus \mathfrak{so}(4) &\sim \mathfrak{so}(12) \oplus \mathfrak{su}(2)_H \oplus \mathfrak{su}(2)' \\ &= \mathfrak{usp}(6) \oplus \mathfrak{su}(2)_{\mathfrak{so}(12)} \oplus \mathbf{3} \times \mathbf{14} \oplus \mathfrak{su}(2)_{(H)} \oplus \mathfrak{su}(2)'; \end{aligned} \quad (4.27)$$

$$\begin{aligned} SO(12) \times SO(4) &\sim SO(12) \times SU(2)_H \times SU(2)' \\ &\supset USp(6) \times SU(2)_{SO(12)} \times SU(2)_H \times SU(2)'; \end{aligned} \quad (4.28)$$

$$\begin{aligned} (\mathbf{66}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}) &= (\mathbf{21}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}) \\ &\quad + (\mathbf{14}, \mathbf{3}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3}). \end{aligned} \quad (4.29)$$

As mentioned,  $SU(2)_H \times SU(2)' \sim SO(4)$  (3.31) and  $SU(2)' \sim USp(2)$  are the  $\mathcal{R}$ -symmetry of the magical quaternionic theory in  $D = 3$  and  $D = 5$ , respectively.  $SO(12)$  and  $USp(6)$  are to be interpreted as the corresponding *Clifford vacuum symmetry* in  $D = 3$  and  $D = 5$ , encoding the further degeneracy due matter vector multiplets.

It holds that

$$\mathfrak{su}(2)' \cap \mathfrak{e}_{7(-5)} = \emptyset \Rightarrow \mathfrak{su}(2)' \cap \mathfrak{su}(2)_J = \emptyset; \quad \mathfrak{su}(2)' \cap \mathfrak{sl}(3, \mathbb{R}) = \emptyset; \quad (4.30)$$

$$\mathfrak{su}(2)_{\mathfrak{so}(12)} \oplus \mathfrak{su}(2)_H \not\subset \mathfrak{sl}(3, \mathbb{R}). \quad (4.31)$$

The  $D = 5$  Ehlers Lie algebra  $\mathfrak{sl}(3, \mathbb{R})$  admits the massless spin algebra  $\mathfrak{su}(2)_J$  as maximal compact subalgebra.

Clearly, the branchings (4.10) and (4.11) hold in this case, as well; however, now  $\frac{E_{7(-5)}}{SO(12) \times SU(2)_H}$ , whose generators sit into the  $(\mathbf{32}', \mathbf{2})$  of  $SO(12) \times SU(2)_H$ , pertains to the bosonic sector of  $\mathcal{N} = 4$ ,  $D = 3$   $\mathfrak{J}_3^{\mathbb{H}}$ -related magical supergravity. Analogously,  $\frac{SU^*(6)}{USp(6)}$  is now to be considered as the rank-2 real special symmetric scalar manifold of the corresponding magical Maxwell-Einstein theory in  $D = 5$ .

Thus, under (3.33)-(3.34), it is worth considering also the following branching:

$$\begin{aligned} E_{7(-133)} \times SU(2)_H \times SU(2)' &\supset F_{4(-52)} \times SU(2)_{E_7} \times SU(2)_H \times SU(2)'; \\ (\mathbf{56}, \mathbf{2}, \mathbf{1}) &= (\mathbf{1}, \mathbf{4}, \mathbf{2}, \mathbf{1}) + (\mathbf{26}, \mathbf{2}, \mathbf{2}, \mathbf{1}). \end{aligned} \quad (4.32)$$

As within the supersymmetrization with 24 local supersymmetries, also in this case the massless  $D = 5$  spin group can be identified with the diagonal  $SU(2)$  into  $SU(2)_{SO(12)} \times SU(2)_H$ , as given by

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<sup>10</sup>This is a *unified*  $\mathcal{N} = 2$  theory : all vectors sit in the  $\mathbf{15}$  irrep. of  $SU^*(6)$ .



(4.13). Thus, (4.28) can be completed to the following chain:

$$\begin{aligned}
SO(12) \times SO(4) &\sim SO(12) \times SU(2)_H \times SU(2)' \\
&\supset USp(6) \times SU(2)_{SO(12)} \times SU(2)_H \times SU(2)' \\
&\supset USp(6) \times SU(2)_J \times SU(2)'; \tag{4.33}
\end{aligned}$$

$$\begin{aligned}
(\mathbf{66}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}) &= (\mathbf{21}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}) \\
&\quad + (\mathbf{14}, \mathbf{3}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3}) \\
&= (\mathbf{21}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{14} + \mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}); \tag{4.34}
\end{aligned}$$

$$\begin{aligned}
(\mathbf{32}', \mathbf{2}, \mathbf{1}) &= (\mathbf{14}, \mathbf{2}, \mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{4}, \mathbf{2}, \mathbf{1}) \\
&= (\mathbf{14}, \mathbf{3}, \mathbf{1}) + (\mathbf{14}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{5}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}). \tag{4.35}
\end{aligned}$$

The decomposition (4.35) corresponds to the massless bosonic spectrum of  $\mathcal{N} = 2$ ,  $D = 5$  magical quaternionic supergravity : consistent with the fact that this theory is the “bosonic twin” of the  $\mathcal{N} = 6$ ,  $D = 5$  “pure” supergravity, they share the very same bosonic spectrum (64 states): 1 graviton, 14 + 1 vectors (in the  $\mathcal{N} = 2$  case, this splitting distinguishes between the graviphoton and the 14 vectors from the vector multiplets), and 14 real scalar fields (in the  $\mathcal{N} = 2$  case, all belonging to the 14 vector multiplets). Such states fit into

$$\mathcal{N} = 6 \text{ (24 susys)} : (\mathbf{32}', \mathbf{2}) \text{ of } SO(12) \times SU(2)_{(H)}; \tag{4.36}$$

$$\mathcal{N} = 2 \text{ (8 susys)} : (\mathbf{32}', \mathbf{2}, \mathbf{1}) \text{ of } SO(12) \times SO(4) \sim SO(12) \times SU(2)_H \times SU(2)'. \tag{4.37}$$

However, the two theories have different fermionic sector; thus, consistently, the massless fermionic spectrum of  $\mathcal{N} = 2$ ,  $D = 5$  magical quaternionic supergravity is not given by  $(\mathbf{32}, \mathbf{2}, \mathbf{1})$ , but rather by  $(\mathbf{32}', \mathbf{1}, \mathbf{2})$ , of  $SO(12) \times SU(2)_H \times SU(2)'$ . Indeed, under (4.33), such an irrep. decomposes as follows:

$$(\mathbf{32}', \mathbf{1}, \mathbf{2}) = (\mathbf{14}, \mathbf{2}, \mathbf{1}, \mathbf{2}) + (\mathbf{1}, \mathbf{4}, \mathbf{1}, \mathbf{2}) = (\mathbf{14}, \mathbf{2}, \mathbf{2}) + (\mathbf{1}, \mathbf{4}, \mathbf{2}), \tag{4.38}$$

thus corresponding to 14  $SU(2)'$ -doublets of gauginos (from the 14 vector multiplets), and 1  $SU(2)'$ -doublet of gravitinos. Thus, at the level of massless spectrum, in the minimal interpretation the action of supersymmetry amounts to the following exchange of irreps.<sup>11</sup>:

$$SO(12) \times SU(2)_H \times SU(2)' : (\mathbf{32}', \mathbf{1}, \mathbf{2}) \xleftrightarrow[F]{B} (\mathbf{32}', \mathbf{2}, \mathbf{1}), \tag{4.39}$$

to be contrasted with its analogue (4.18), holding in presence of 24 local supersymmetries. Note that, consistently, bosons are  $\mathcal{R}$ -symmetry  $SU(2)'$ -singlets, whereas fermions fit into  $SU(2)'$ -doublets.

*Mutatis mutandis*, the very same considerations made in Subsec. 4.1 (in particular, the ones at points 4 and 5 therein) also hold in this case. Due to the 8-supersymmetries interpretation, from the remarks made at point 4 of Subsec. 3.2, the  $D = 3$  manifold  $\frac{E_{7(-5)}}{SO(12) \times SU(2)_H}$  can here be regarded as the following  $c$ -map image [43]:

$$\frac{SO^*(12)}{SU(6) \times U(1)}_{D=4} \xrightarrow{c} \frac{E_{7(-5)}}{SO(12) \times SU(2)_H}_{D=4}, \tag{4.40}$$

where the coset on the l.h.s. is the symmetric rank-3 special Kähler vector multiplets' scalar manifold of the magical  $\mathcal{N} = 2$ ,  $D = 4$  quaternionic supergravity.

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<sup>11</sup>Consistently with the branching properties

$$\begin{aligned}
SO(12) &\supset USp(6) \times SU(2)_{SO(12)}, \\
\mathbf{32} &= (\mathbf{6}, \mathbf{3}) + (\mathbf{14}', \mathbf{1}),
\end{aligned}$$

the irrep.  $(\mathbf{32}, \mathbf{2}, \mathbf{1})$  of  $SO(12) \times SU(2)_H \times SU(2)'$  does *not* occur as (massless) bosonic or fermionic representation in the  $D = 5$  theory.

## 5 Simple Jordan Pair Embeddings

In the present Section, we list and briefly analyze the relevant *non-compact* real forms (2.5) of the *compact Jordan pair* embeddings (2.7) (listed in Table 1) [9] pertaining to *simple* Euclidean rank-3 Jordan algebras.

We will thus briefly reconsider the cases  $q = 8 \mathfrak{J}_3^{\mathbb{O}_s}$  and  $\mathfrak{J}_3^{\mathbb{O}}$ , and  $q = 4 \mathfrak{J}_3^{\mathbb{H}}$  (recall Footnote 1 for the cases  $q = -1$  and  $q = -4/3$ ), but we will not mention the peculiar (non-simple but triality-symmetric) case  $q = 0$  ( $J_3 = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ ), which deserves a separate treatment, given in Subsubsec. 6.1.1.

### 5.1 $q = 8$

- $\mathfrak{J}_3^{\mathbb{O}}$  ( $\mathfrak{qconf}(\mathfrak{J}_3^{\mathbb{O}}) = \mathfrak{e}_{8(-24)}; \mathfrak{str}_0(\mathfrak{J}_3^{\mathbb{O}}) = \mathfrak{e}_{6(-26)} \sim \mathfrak{sl}(3, \mathbb{O})$ ):

$$\mathfrak{e}_{8(-24)} = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{e}_{6(-26)} \oplus \mathbf{3} \times \mathbf{27} \oplus \mathbf{3}' \times \mathbf{27}'; \quad (5.1)$$

$$E_{8(-24)} \supset SL(3, \mathbb{R}) \times E_{6(-26)}; \quad (5.2)$$

$$\mathbf{248} = (\mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{78}) + (\mathbf{3}, \mathbf{27}) + (\mathbf{3}', \mathbf{27}'); \quad (5.3)$$

$$\mathfrak{J}_3^{\mathbb{O}} : E_{6(-26)} \supset F_{4(-52)} : \mathbf{27} = \mathbf{26} + \mathbf{1}. \quad (5.4)$$

In (5.4),  $\mathbf{27}$  and  $\mathbf{26}$  respectively are the fundamental irreps. of  $E_{6(-26)}$  and of its maximal compact subgroup  $F_{4(-52)}$ . Physical (supergravity) interpretation : minimal theory (8 supersymmetries, with  $D$ -independent hypersector). See Subsec. 3.2.

- $\mathfrak{J}_3^{\mathbb{O}_s}$  ( $\mathfrak{qconf}(\mathfrak{J}_3^{\mathbb{O}_s}) = \mathfrak{e}_{8(8)}; \mathfrak{str}_0(\mathfrak{J}_3^{\mathbb{O}_s}) = \mathfrak{e}_{6(6)} \sim \mathfrak{sl}(3, \mathbb{O}_s)$ ):

$$\mathfrak{e}_{8(8)} = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{e}_{6(6)} \oplus \mathbf{3} \times \mathbf{27} \oplus \mathbf{3}' \times \mathbf{27}'; \quad (5.5)$$

$$E_{8(8)} \supset SL(3, \mathbb{R}) \times E_{6(6)}; \quad (5.6)$$

$$\mathbf{248} = (\mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{78}) + (\mathbf{3}, \mathbf{27}) + (\mathbf{3}', \mathbf{27}'); \quad (5.7)$$

$$\mathfrak{J}_3^{\mathbb{O}_s} : E_{6(6)} \supset USp(8) : \mathbf{27} = \mathbf{27}. \quad (5.8)$$

In (5.8),  $\mathbf{27}$  is the fundamental irrep. of  $E_{6(6)}$ , which becomes the rank-2 antisymmetric skew-traceless irrep. of its maximal compact subgroup  $USp(8)$ . Physical (supergravity) interpretation : maximal theory (32 supersymmetries, without  $D$ -independent hypersector). See Subsec. 3.1.

### 5.2 $q = 4$

- $\mathfrak{J}_3^{\mathbb{H}}$  ( $\mathfrak{qconf}(\mathfrak{J}_3^{\mathbb{H}}) = \mathfrak{e}_{7(-5)}; \mathfrak{str}_0(\mathfrak{J}_3^{\mathbb{H}}) = \mathfrak{su}^*(6) \sim \mathfrak{sl}(3, \mathbb{H})$ ;  $\mathbf{14}$  is the rank-2 antisymmetric skew-traceless of  $USp(6)$ ):

$$\mathfrak{e}_{7(-5)} = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{su}^*(6) \oplus \mathbf{3} \times \mathbf{15}' \oplus \mathbf{3}' \times \mathbf{15}; \quad (5.9)$$

$$E_{7(-5)} \supset SL(3, \mathbb{R}) \times SU^*(6); \quad (5.10)$$

$$\mathbf{133} = (\mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{35}) + (\mathbf{3}, \mathbf{15}') + (\mathbf{3}', \mathbf{15}); \quad (5.11)$$

$$\mathfrak{J}_3^{\mathbb{H}} : SU^*(6) \supset USp(6) : \mathbf{15}' = \mathbf{14} + \mathbf{1}. \quad (5.12)$$

In (5.12),  $\mathbf{15}'$  and  $\mathbf{14}$  respectively are the (contravariant) rank-2 antisymmetric irrep. of  $SU^*(6)$  and the rank-2 antisymmetric skew-traceless irrep. of its maximal compact subgroup  $USp(6)$ . Physical (supergravity) interpretation : *either* minimal theory (8 supersymmetries, with  $D$ -independent hypersector), *or* “pure” theory with 24 supersymmetries (without  $D$ -independent hypersector) : this is indeed example of a pair of “(bosonic) twin” theories; see Sec. 4.

- $\mathfrak{J}_3^{\mathbb{H}_s} (\text{qconf}(\mathfrak{J}_3^{\mathbb{H}_s}) = \mathfrak{e}_{7(7)} = \text{conf}(\mathfrak{J}_3^{\mathbb{O}_s}); \text{str}_0(\mathfrak{J}_3^{\mathbb{H}_s}) = \mathfrak{sl}(6, \mathbb{R}) \sim \mathfrak{sl}(3, \mathbb{H}_s))$ :

$$\mathfrak{e}_{7(7)} = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(6, \mathbb{R}) \oplus \mathbf{3} \times \mathbf{15}' \oplus \mathbf{3}' \times \mathbf{15}; \quad (5.13)$$

$$E_{7(7)} \supset SL(3, \mathbb{R}) \times SL(6, \mathbb{R}); \quad (5.14)$$

$$\mathbf{133} = (\mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{35}) + (\mathbf{3}, \mathbf{15}') + (\mathbf{3}', \mathbf{15}); \quad (5.15)$$

$$\mathfrak{J}_3^{\mathbb{H}_s} : SL(6, \mathbb{R}) \supset SO(6) : \mathbf{15}' = \mathbf{15}. \quad (5.16)$$

In (5.16),  $\mathbf{15}'$  is the (contravariant) rank-2 antisymmetric of  $SL(6, \mathbb{R})$ , which becomes the adjoint of its maximal compact subgroup  $SO(6) \sim SU(4)$ . Physical interpretation : *non-supersymmetric* theory ( $D$ -independent hypersector irrelevant); in fact,  $E_{7(7)}$  can be the global symmetry of a non-linear scalar sigma model coupled to gravity in  $D = 3$  dimensions (see *e.g.* [48]).

### 5.3 $q = 2$

- $\mathfrak{J}_3^{\mathbb{C}} (\text{qconf}(\mathfrak{J}_3^{\mathbb{C}}) = \mathfrak{e}_{6(2)}; \text{str}_0(\mathfrak{J}_3^{\mathbb{C}}) = \mathfrak{sl}(3, \mathbb{C}))$ :

$$\mathfrak{e}_{6(2)} = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathbf{3} \times (\mathbf{3}, \overline{\mathbf{3}}) \oplus \mathbf{3}' \times (\overline{\mathbf{3}}, \mathbf{3}); \quad (5.17)$$

$$E_{6(2)} \supset SL(3, \mathbb{R}) \times SL(3, \mathbb{C}); \quad (5.18)$$

$$\mathbf{78} = (\mathbf{8}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{8}) + (\mathbf{3}, \mathbf{3}, \overline{\mathbf{3}}) + (\mathbf{3}', \overline{\mathbf{3}}, \mathbf{3}); \quad (5.19)$$

$$\mathfrak{J}_3^{\mathbb{C}} : SL(3, \mathbb{C}) \supset SU(3) : (\mathbf{3}, \overline{\mathbf{3}}) = \mathbf{8} + \mathbf{1}. \quad (5.20)$$

Physical (supergravity) interpretation : minimal theory (8 supersymmetries, with  $D$ -independent hypersector). It is worth recalling here that, from the theory of extremal black hole attractors, another maximal non-symmetric embedding is known (see *e.g.* App. of [49]):

$$E_{6(2)} \supset SU(2, 1) \times SU(2, 1) \times SU(2, 1). \quad (5.21)$$

- $\mathfrak{J}_3^{\mathbb{C}_s} (\text{qconf}(\mathfrak{J}_3^{\mathbb{C}_s}) = \mathfrak{e}_{6(6)} = \text{str}_0(\mathfrak{J}_3^{\mathbb{O}_s}); \text{str}_0(\mathfrak{J}_3^{\mathbb{C}_s}) = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R}) \sim \mathfrak{sl}(3, \mathbb{C}_s))$ :

$$\mathfrak{e}_{6(6)} = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})_I \oplus \mathfrak{sl}(3, \mathbb{R})_{II} \oplus \mathbf{3} \times (\mathbf{3}, \mathbf{3}') \oplus \mathbf{3}' \times (\mathbf{3}', \mathbf{3}); \quad (5.22)$$

$$E_{6(6)} \supset SL(3, \mathbb{R}) \times SL(3, \mathbb{R})_I \times SL(3, \mathbb{R})_{II}; \quad (5.23)$$

$$\mathbf{78} = (\mathbf{8}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{8}) + (\mathbf{3}, \mathbf{3}, \mathbf{3}') + (\mathbf{3}', \mathbf{3}', \mathbf{3}); \quad (5.24)$$

$$\mathfrak{J}_3^{\mathbb{C}_s} : SL(3, \mathbb{R})_I \times SL(3, \mathbb{R})_{II} \supset SO(3) \times SO(3) : (\mathbf{3}, \mathbf{3}') = (\mathbf{3}, \mathbf{3}). \quad (5.25)$$

Note that (5.24) does not give rise to a triality-symmetric decomposition. Moreover, (5.25) is a (double) maximal symmetric principal embedding, of the same kind of the maximal enhancement of the  $D = 5$  spin group  $SU(2)_J$  into  $SL(3, \mathbb{R})_{\text{Ehlers}}$  (*cfr.* *e.g.* (3.16)). Physical interpretation : *non-supersymmetric* theory ( $D$ -independent hypersector irrelevant); in fact,  $E_{6(6)}$  can be a global symmetry of a non-linear scalar sigma model coupled to  $D = 3$  gravity (see *e.g.* [48]).

### 5.4 $q = 1$

- $\mathfrak{J}_3^{\mathbb{R}} (\text{qconf}(\mathfrak{J}_3^{\mathbb{R}}) = \mathfrak{f}_{4(4)}; \text{str}_0(\mathfrak{J}_3^{\mathbb{R}}) = \mathfrak{sl}(3, \mathbb{R}))$ :

$$\mathfrak{f}_{4(4)} = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})_I \oplus \mathbf{3} \times \mathbf{6}' \oplus \mathbf{3}' \times \mathbf{6}; \quad (5.26)$$

$$F_{4(4)} \supset SL(3, \mathbb{R}) \times SL(3, \mathbb{R})_I; \quad (5.27)$$

$$\mathbf{52} = (\mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{8}) + (\mathbf{3}, \mathbf{6}') + (\mathbf{3}', \mathbf{6}); \quad (5.28)$$

$$\mathfrak{J}_3^{\mathbb{R}} : SL(3, \mathbb{R})_I \supset SO(3) : \mathbf{6}' = \mathbf{5} + \mathbf{1}. \quad (5.29)$$

In (5.29),  $\mathbf{6}'$  is the (contravariant) rank-2 symmetric of  $SL(3, \mathbb{R})_I$ . As for  $\mathfrak{J}_3^{\mathbb{C}_s}$ , (5.29) is a maximal symmetric principal embedding, of the same kind of the maximal enhancement of the  $D = 5$  massless spin group  $SU(2)_J$  into  $SL(3, \mathbb{R})_{\text{Ehlers}}$  (*cfr. e.g.* (3.16)). Physical (supergravity) interpretation : minimal theory (8 supersymmetries, with  $D$ -independent hypersector). It is worth recalling here that, from the theory of attractors in  $D = 5$ ,  $\mathfrak{J}_3^{\mathbb{C}}$ -related magical supergravity, another maximal non-symmetric embedding is known (see *e.g.* App. of [49]):

$$F_{4(4)} \supset SU(2, 1) \times SU(2, 1). \quad (5.30)$$

## 5.5 $q = -2/3$

- $\mathbb{R}(\mathfrak{qconf}(\mathbb{R}) = \mathfrak{g}_{2(2)}; \mathfrak{str}_0(\mathbb{R}) = \emptyset$ , and thus no non-trivial Jordan algebra representation):

$$\mathfrak{g}_{2(2)} = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathbf{3} \oplus \mathbf{3}'; \quad (5.31)$$

$$G_{2(2)} \supset SL(3, \mathbb{R}); \quad (5.32)$$

$$\mathbf{14} = \mathbf{8} + \mathbf{3} + \mathbf{3}'; \quad (5.33)$$

$$\mathbb{R} : \text{Id} : \mathbf{1} = \mathbf{1}. \quad (5.34)$$

For more on the maximal non-symmetric embedding (5.32)-(5.33), see *e.g.* App. [33], and Refs. therein, as well as [10]). Physical (supergravity) interpretation : minimal theory (8 supersymmetries, with  $D$ -independent hypersector), named  $T^3$  model in  $D = 4$ . It is worth recalling here that, from the theory of  $c$ -map in supergravity (namely, from the *universal hypermultiplet* as  $c$ -map of “pure”  $\mathcal{N} = 2$ ,  $D = 4$  supergravity [43]), another maximal non-symmetric embedding is known:

$$G_{2(2)} \supset SU(2, 1). \quad (5.35)$$

## 6 Jordan Pairs : the Semi-Simple Case

We are now going to extend the treatment of *Jordan pair* embeddings, introduced in Sec. 2 for *simple* rank-3 Euclidean Jordan algebras, to *semi-simple* rank-3 Euclidean Jordan algebras, having the following structure [13]:

$$\mathfrak{J}_3^{m,n} \equiv \mathbb{R} \oplus \mathbf{\Gamma}_{m-1,n-1}, \quad (6.1)$$

where  $\mathbf{\Gamma}_{m-1,n-1}$  is the *simple* rank-2 Euclidean Jordan algebra given by the Clifford algebra of  $O(m-1, n-1)$ , with

$$\mathfrak{str}_0(\mathbf{\Gamma}_{m-1,n-1}) = \mathfrak{so}(m-1, n-1). \quad (6.2)$$

The relevant cases for supergravity corresponds to:

- $m = 2$ , pertaining to an infinite sequence of models with 8 local supersymmetries (coupled to  $n$  vector multiplets, namely 1 dilatonic and  $n-1$  non-dilatonic vector multiplets, in  $D = 5$ ), based on

$$8 \text{ susys} : \mathfrak{J}_3^{2,n} \equiv \mathbb{R} \oplus \mathbf{\Gamma}_{1,n-1}; \quad (6.3)$$

- $m = 6$ , pertaining to half-maximal supergravity (16 supersymmetries), coupled to  $n-1$  matter (vector) multiplets in  $D = 5$ , based on

$$16 \text{ susys} : \mathfrak{J}_3^{6,n} \equiv \mathbb{R} \oplus \mathbf{\Gamma}_{5,n-1}. \quad (6.4)$$

In general, it holds that

$$\mathfrak{L}(\mathfrak{J}_3^{m,n}) \equiv \mathfrak{qconf}(\mathfrak{J}_3^{m,n}) = \mathfrak{so}(m+2, n+2); \quad (6.5)$$

$$\mathfrak{conf}(\mathfrak{J}_3^{m,n}) = \mathfrak{aut}(\mathfrak{J}_3^{m,n}) = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(m, n); \quad (6.6)$$

$$\mathfrak{str}_0(\mathfrak{J}_3^{m,n}) = \mathfrak{so}(1, 1) \oplus \mathfrak{so}(m-1, n-1) = \mathfrak{so}(1, 1) \oplus \mathfrak{str}_0(\mathbf{\Gamma}_{m-1,n-1}), \quad (6.7)$$

where  $\mathfrak{F}(\mathfrak{J}_3^{m,n})$  denotes the *Freudenthal triple system* constructed over  $\mathfrak{J}_3^{m,n}$  (see *e.g.* [20, 50], and Refs. therein).

Furthermore, for (6.1) one can define an “effective” parameter  $q_{eff}(m, n)$  as follows:

$$3q_{eff}(m, n) + 4 = m + n \Leftrightarrow q_{eff}(m, n) = \frac{m + n - 4}{3}, \quad (6.8)$$

such that the “effective” dimension of  $\mathfrak{J}_3^{m,n}$  relevant for the *semi-simple* generalization of the *Jordan pair* embeddings (2.5) and (2.7) reads

$$3q_{eff}(m, n) + 3 = m + n - 1. \quad (6.9)$$

In the following treatment, we will focus on the aforementioned cases  $m = 2$  and  $m = 6$ , relevant for supergravity (a general treatment for (6.1) can be given by a straightforward generalization<sup>12</sup>).

### 6.1 $\mathfrak{J}_3^{2,n} \equiv \mathbb{R} \oplus \mathbf{\Gamma}_{1,n-1}$

An important difference between the *simple* Jordan algebras treated in Secs. 2-5 and the *semi-simple* Jordan algebras (6.1) is the fact that  $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{str}_0(\mathfrak{J}_3^{m,n})$  is *not* maximally embedded into  $\mathfrak{L}(\mathfrak{J}_3^{m,n})$ , but rather it can be embedded by a *two-step* chain of maximal symmetric embeddings.

In the case  $m = 2$  under consideration, this chain reads as follows ( $\mathfrak{so}(3, 3) \sim \mathfrak{sl}(4, \mathbb{R})$ ):

$$\begin{aligned} \mathfrak{so}(4, n+2) &\supset \mathfrak{so}(3, 3) \oplus \mathfrak{so}(1, n-1) \oplus \mathbf{6} \times \mathbf{n} \\ &\supset \mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{so}(1, n-1) \oplus \mathfrak{so}(1, 1) \oplus \mathbf{3} \times (\mathbf{n}_2 + \mathbf{1}_{-4}) \oplus \mathbf{3}' \times (\mathbf{n}_{-2} + \mathbf{1}_4), \end{aligned} \quad (6.10)$$

or, at group level:

$$\begin{aligned} SO(4, n+2) &\supset SO(3, 3) \times SO(1, n-1) \\ &\supset SL(3, \mathbb{R}) \times SO(1, n-1) \times SO(1, 1); \end{aligned} \quad (6.11)$$

$$\begin{aligned} \mathbf{Adj}_{SO(4,n+2)} &= \mathbf{Adj}_{SO(3,3)} + \mathbf{Adj}_{SO(1,n-1)} + (\mathbf{6}, \mathbf{n}) \\ &= \mathbf{Adj}_{SL(3,\mathbb{R})} + \mathbf{Adj}_{SO(1,1)} + \mathbf{Adj}_{SO(1,n-1)} \\ &\quad + (\mathbf{3}, \mathbf{n}_2 + \mathbf{1}_{-4}) + (\mathbf{3}', \mathbf{n}_{-2} + \mathbf{1}_4), \end{aligned} \quad (6.12)$$

where the subscripts denote  $SO(1, 1)$ -weights.

Note that, according to (6.8) and (6.9), the “effective” dimension of  $\mathfrak{J}_3^{2,n}$  is  $n + 1$ , and the corresponding representation is *reducible* with respect to  $Str_0(\mathfrak{J}_3^{2,n}) = SO(1, 1) \times SO(1, n-1)$ , as given by (6.12):

$$\mathbf{n} + \mathbf{1} = \mathbf{n}_2 + \mathbf{1}_{-4}. \quad (6.13)$$

Thus, the  $\mathfrak{J}_3^{2,n}$ -related  $\mathcal{N} = 2$  theory in  $D = 5$  is *not unified* [40]; this is another difference with respect to simple Jordan algebras, in which  $\mathfrak{J}_3^q$  fits into an *irreducible* representation of  $Str_0(\mathfrak{J}_3^q)$  itself, and therefore the corresponding  $D = 5$  theory is *unified*. In (6.13) (modulo redefinitions of the  $SO(1, 1)$  weights),  $\mathbf{n}_2$  corresponds to the graviphoton and the  $n - 1$  matter vectors (respectively with positive and negative signature in  $SO(1, n-1)$ ), whereas  $\mathbf{1}_{-4}$  pertains to the vector from the dilatonic vector multiplet.

The second line of (6.10) provides the extension of (2.5) to  $\mathfrak{J}_3^{2,n}$  (6.3). Its compact counterpart (which extends (2.7), and thus the results of [9]) reads

$$\begin{aligned} \mathfrak{so}(n+6) &\supset \mathfrak{so}(6) \oplus \mathfrak{so}(n) \oplus \mathbf{6} \times \mathbf{n} \\ &\supset \mathfrak{su}(3) \oplus \mathfrak{so}(n) \oplus \mathfrak{u}(1) \oplus \mathbf{3} \times (\mathbf{n}_2 + \mathbf{1}_{-4}) \oplus \overline{\mathbf{3}} \times (\mathbf{n}_{-2} + \mathbf{1}_4), \end{aligned} \quad (6.14)$$

<sup>12</sup>It is easily realized that  $\mathfrak{J}_3^{m,n} \sim \mathfrak{J}_3^{n,m}$  as vector space isomorphism. Actually, for  $(m, n) = (2, 6)$ , this entails a pair of “(bosonic) twin” theories, whose treatment in  $D = 5$  in terms of *Jordan pairs* is given in Sec. 7.

with subscripts here denoting  $U(1)$ -charges ( $\mathfrak{so}(6) \sim su(4)$ ). (Suitable real, non-compact forms of) orthogonal Lie algebras can be characterized as quasi-conformal algebras of suitable *semi-simple* Euclidean Jordan algebras of rank 3 [50].

At group level, the algebraic decompositions (6.10) and (6.14) respectively correspond to the second line of (6.11), which can be summarized by the non-maximal non-symmetric embedding

$$QConf\left(\mathfrak{J}_3^{2,n}\right) \supset SL(3, \mathbb{R}) \times Str_0\left(\mathfrak{J}_3^{2,n}\right), \quad (6.15)$$

and by its compact counterpart:

$$SO(n+6) \supset SU(3) \times SO(n) \Leftrightarrow QConf_c\left(\mathfrak{J}_3^{2,n}\right) \supset SU(3) \times Str_{0,c}\left(\mathfrak{J}_3^{2,n}\right). \quad (6.16)$$

As mentioned,  $\mathfrak{J}_3^{2,n}$  is related to minimal supergravity (8 supersymmetries): therefore, matter coupling is allowed, in terms of two types of matter multiplets, namely vector and hyper multiplets, and a  $D$ -independent hypermultiplet sector must be considered.

The  $mcs$  of  $\mathfrak{qconf}\left(\mathfrak{J}_3^{2,n}\right) = \mathfrak{so}(4, n+2)$  and  $\mathfrak{str}_0\left(\mathfrak{J}_3^{2,n}\right) = \mathfrak{so}(1, 1) \oplus \mathfrak{so}(1, n-1)$  respectively read<sup>13</sup>

$$mcs(\mathfrak{so}(4, n+2)) = \mathfrak{so}(n+2) \oplus \mathfrak{so}(4) \sim \mathfrak{so}(n+2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)_H; \quad (6.17)$$

$$mcs(\mathfrak{so}(1, 1) \oplus \mathfrak{so}(1, n-1)) = \mathfrak{so}(n-1), \quad (6.18)$$

corresponding to the following maximal symmetric embeddings at group level:

$$SO(4, n+2) \supset SO(n+2) \times SO(4) \sim SO(n+2) \times SU(2) \times SU(2)_H; \quad (6.19)$$

$$\begin{aligned} \mathbf{Adj}_{SO(4, n+2)} &= \mathbf{Adj}_{SO(n+2)} + \mathbf{Adj}_{SO(4)} + (\mathbf{n} + \mathbf{2}, \mathbf{4}) \\ &= \mathbf{Adj}_{SO(n+2)} + \mathbf{Adj}_{SU(2)} + \mathbf{Adj}_{SU(2)_H} + (\mathbf{n} + \mathbf{2}, \mathbf{2}, \mathbf{2}); \end{aligned} \quad (6.20)$$

$$SO(1, 1) \times SO(1, n-1) \supset SO(n-1); \quad (6.21)$$

$$\mathbf{1}_0 + \mathbf{Adj}_{SO(1, n-1), 0} = \mathbf{1} + \mathbf{Adj}_{SO(n-1)} + (\mathbf{n} - \mathbf{1}). \quad (6.22)$$

However, the relevant maximal embedding must include the  $\mathfrak{su}(2)'$  algebra from the  $D$ -independent hypersector, as well:

$$\begin{aligned} \mathfrak{so}(n+2) \oplus \mathfrak{so}(4) \oplus \mathfrak{su}(2)' &\sim \mathfrak{so}(n+2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)_H \oplus \mathfrak{su}(2)' \\ &= \mathfrak{so}(n-1) \oplus \mathfrak{su}(2)_{\mathfrak{so}(n+2)} \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)_H \oplus \mathfrak{su}(2)' \\ &\quad \oplus (\mathbf{n} - \mathbf{1}) \times \mathbf{3} \times \mathbf{1} \times \mathbf{1} \times \mathbf{1}; \end{aligned} \quad (6.23)$$

$$\begin{aligned} SO(n+2) \times SO(4) \times SU(2)' &\sim SO(n+2) \times SU(2) \times SU(2)_H \times SU(2)' \\ &\supset SO(n-1) \times SU(2)_{SO(n+2)} \\ &\quad \times SU(2) \times SU(2)_H \times SU(2)'; \end{aligned} \quad (6.24)$$

$$\begin{aligned} \mathbf{Adj}_{SO(n+2)} + \mathbf{Adj}_{SO(4)} + \mathbf{Adj}_{SU(2)'} &= \mathbf{Adj}_{SO(n+2)} + \mathbf{Adj}_{SU(2)} + \mathbf{Adj}_{SU(2)_H} + \mathbf{Adj}_{SU(2)'} \\ &= \mathbf{Adj}_{SO(n-1)} + \mathbf{Adj}_{SU(2)_{SO(n+2)}} \\ &\quad + \mathbf{Adj}_{SU(2)} + \mathbf{Adj}_{SU(2)_H} + \mathbf{Adj}_{SU(2)'} \\ &\quad + (\mathbf{n} - \mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1}). \end{aligned} \quad (6.25)$$

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<sup>13</sup>The presence of an “extra” commuting  $SU(2)$  in (6.17) and (6.19) can ultimately be traced back to the fact that for  $m=2$   $SO(m+2) = SO(4) \sim SU(2) \times SU(2)_H$ .

By  $SU(2)_{SO(n+2)}$ , we here denote the group commuting with  $SO(n-1)$  in the maximal symmetric embedding

$$SO(n+2) \supset SO(n-1) \times SO(3) \sim SO(n-1) \times SU(2)_{SO(n+2)}, \quad (6.26)$$

determining (6.23). Note that, differently from its analogue for *simple* rank-3 Euclidean Jordan algebras, the maximal embedding (6.23)-(6.24) is symmetric.

Moreover, it is worth pointing out that the  $\mathcal{R}$ -symmetry of  $\mathcal{N} = 4$ ,  $D = 3$   $\mathfrak{J}_3^{2,n}$ -related supergravity is consistently enhanced from the quaternionic  $SU(2)_H$  (related to the  $c$ -map [43] of the  $D = 4$  vector multiplets' scalar manifold) to  $SU(2)_H \times SU(2)'$ , as given by (3.31). On the other hand,  $SU(2)' \sim USp(2)$  is the  $\mathcal{R}$ -symmetry of the corresponding  $D = 5$  ( $\mathcal{N} = 2$ ) uplift of the theory.

Clearly, it holds that

$$\mathfrak{su}(2)' \cap \mathfrak{so}(4, n+2) = \emptyset \Rightarrow \mathfrak{su}(2)' \cap \mathfrak{su}(2)_J = \emptyset; \quad \mathfrak{su}(2)' \cap \mathfrak{sl}(3, \mathbb{R}) = \emptyset; \quad (6.27)$$

$$\mathfrak{su}(2)_{\mathfrak{so}(n+2)} \oplus \mathfrak{su}(2)_H \not\subseteq \mathfrak{sl}(3, \mathbb{R}), \quad (6.28)$$

where, as in general, the  $D = 5$  Ehlers Lie algebra  $\mathfrak{sl}(3, \mathbb{R})$  admits the massless spin algebra  $\mathfrak{su}(2)_J$  as maximal compact subalgebra.

From the embedding (6.17), (6.19) and (6.20),  $(\mathbf{n} + \mathbf{2}, \mathbf{2}, \mathbf{2})$  is the tri-fundamental irrep. of  $SO(n+2) \times SU(2) \times SU(2)_H$ , in which the generators of the rank-4 symmetric quaternionic scalar manifold  $\frac{SO(4, n+2)}{SO(n+2) \times SU(2) \times SU(2)_H}$  of  $\mathcal{N} = 4$ ,  $D = 3$   $\mathfrak{J}_3^{2,n}$ -related supergravity sit. Furthermore, from the embedding (6.18), (6.21) and (6.22),  $\mathbf{1} + (\mathbf{n} - \mathbf{1})$  is the (singlet + fundamental) irrep. of  $SO(n-1)$ , in which the generators of the rank-2 symmetric real special scalar manifold  $SO(1, 1) \times \frac{SO(1, n-1)}{SO(n-1)}$  of the corresponding theory in  $D = 5$  sit. Thus, under (6.23)-(6.24), it is worth considering also the following branching:

$$\begin{aligned} SO(n+2) \times SO(4) \times SU(2)' &\supset SO(n-1) \times SU(2)_{SO(n+2)} \times SU(2) \times SU(2)_H \times SU(2)'; \\ (\mathbf{n} + \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1}) &= (\mathbf{n} - \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{2}, \mathbf{2}, \mathbf{1}). \end{aligned} \quad (6.29)$$

Some remarks are in order.

1. Differently from the treatment of *simple* rank-3 Euclidean Jordan algebras, in order to identify the  $D = 5$  massless *spin group*  $SU(2)_J$ , a two-step procedure is to be performed: **1.1]** one introduces the diagonal  $SU(2)_I$  into  $SU(2) \times SU(2)_H$ :

$$SU(2)_I \subset_d SU(2) \times SU(2)_H, \quad (6.30)$$

such that (6.29) can be completed to the following chain:

$$\begin{aligned} SO(n+2) \times SO(4) \times SU(2)' &\supset SO(n-1) \times SU(2)_{SO(n+2)} \times SU(2) \times SU(2)_H \times SU(2)' \\ &\supset SO(n-1) \times SU(2)_{SO(n+2)} \times SU(2)_I \times SU(2)'; \end{aligned} \quad (6.31)$$

$$\begin{aligned} (\mathbf{n} + \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1}) &= (\mathbf{n} - \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{2}, \mathbf{2}, \mathbf{1}) \\ &= (\mathbf{n} - \mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{n} - \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}). \end{aligned} \quad (6.32)$$

- 1.2]** Then,  $SU(2)_J$  is identified with the diagonal  $SU(2)_{II}$  into  $SU(2)_{SO(n+2)} \times SU(2)_I$ :

$$SU(2)_J \equiv SU(2)_{II} \subset_d SU(2)_{SO(n+2)} \times SU(2)_I. \quad (6.33)$$

Indeed, the chain (6.31)-(6.32) can be further completed as follows:

$$\begin{aligned}
SO(n+2) \times SO(4) \times SU(2)' &\supset SO(n-1) \times SU(2)_{SO(n+2)} \times SU(2) \times SU(2)_H \times SU(2)' \\
&\supset SO(n-1) \times SU(2)_{SO(n+2)} \times SU(2)_I \times SU(2)' \\
&\supset SO(n-1) \times SU(2)_J \times SU(2)';
\end{aligned} \tag{6.34}$$

$$\begin{aligned}
(\mathbf{n} + \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1}) &= (\mathbf{n} - \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{2}, \mathbf{2}, \mathbf{1}) \\
&= (\mathbf{n} - \mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{n} - \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}) \\
&= (\mathbf{n} - \mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{n} - \mathbf{1}, \mathbf{1}, \mathbf{1}) \\
&\quad + (\mathbf{1}, \mathbf{5}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}).
\end{aligned} \tag{6.35}$$

The decomposition (6.35) corresponds to the massless bosonic spectrum of  $\mathcal{N} = 2$ ,  $D = 5$   $\mathfrak{J}_3^{2,n}$ -related supergravity ( $4(n+2)$  states): 1 graviton and 1 graviphoton (from the gravity multiplet), 1 dilatonic vector and 1 dilaton (from the dilatonic vector multiplet), and  $n-1$  vectors and  $n-1$  (real) scalars from the  $n-1$  non-dilatonic vector multiplets. At the level of massless spectrum, the action of supersymmetry amounts to the following exchange of irreps.:

$$SO(n+2) \times SU(2) \times SU(2)_H \times SU(2)': (\mathbf{n} + \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1})_B \longleftrightarrow (\mathbf{n} + \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{2})_F. \tag{6.36}$$

This can be realized by noticing that, under the chain of maximal symmetric (6.34),  $(\mathbf{n} + \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{2})$  decomposes as follows:

$$\begin{aligned}
(\mathbf{n} + \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{2}) &= (\mathbf{n} - \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2}) + (\mathbf{1}, \mathbf{3}, \mathbf{2}, \mathbf{1}, \mathbf{2}) \\
&= (\mathbf{n} - \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}) + (\mathbf{1}, \mathbf{3}, \mathbf{2}, \mathbf{2}) \\
&= (\mathbf{n} - \mathbf{1}, \mathbf{2}, \mathbf{2}) + (\mathbf{1}, \mathbf{4}, \mathbf{2}) + (\mathbf{1}, \mathbf{2}, \mathbf{2}),
\end{aligned} \tag{6.37}$$

thus reproducing the massless fermionic spectrum of  $\mathcal{N} = 2$ ,  $D = 5$   $\mathfrak{J}_3^{2,n}$ -related supergravity ( $4(n+2)$  states): 1  $SU(2)'$ -doublet of gravitinos, 1  $SU(2)'$ -doublet of dilatonic gauginos, and  $n-1$   $SU(2)'$ -doublets of gauginos from the  $n-1$  non-dilatonic vector multiplets. Note that, consistently, bosons are  $\mathcal{R}$ -symmetry  $SU(2)'$ -singlets, whereas fermions fit into  $SU(2)'$ -doublets.

- As generally holding true also for the *semi-simple* cases,  $SU(2)_J$ , which commutes with  $SO(n-1) \times SU(2)'$  inside  $SO(n+2) \times SU(2) \times SU(2)_H \times SU(2)'$  (cfr. (6.34)), is the Kostant “*principal*”  $SU(2)$  (3.16) into the  $D = 5$  Ehlers  $SL(3, \mathbb{R})$ :

$$SL(3, \mathbb{R}) \cap [SU(2)_{SO(n+2)} \times SU(2) \times SU(2)_H] = SU(2)_J. \tag{6.38}$$

- As a consequence of the chain of maximal symmetric embeddings (6.11) and (6.34), the following (non-maximal, non-symmetric) manifold embedding holds:

$$\frac{SO(4, n+2)}{SO(n+2) \times SU(2) \times SU(2)_H} \supset SO(1, 1) \times \frac{SO(1, n-1)}{SO(n-1)} \times \frac{SL(3, \mathbb{R})}{SU(2)_J}. \tag{6.39}$$

As above, this has the trivial interpretation of embedding of the scalar manifold of the  $D = 5$  theory into the scalar manifold of the corresponding theory reduced to  $D = 3$ .

- In  $\mathcal{N} = 2$ ,  $D = 5$   $\mathfrak{J}_3^{2,n}$ -related supergravity, (3.25)-(3.27) respectively specify to [10]

$$M_{\mathcal{N}=2, \mathfrak{J}_3^{2,n}}^5 \equiv \frac{SO(4, n+2)}{SO(1, 1) \times SO(1, n-1) \times SL(3, \mathbb{R})_{\text{Ehlers}}}; \tag{6.40}$$

$$\widehat{M}_{\mathcal{N}=2, \mathfrak{J}_3^{2,n}}^5 \equiv \frac{SO(4) \times SO(n+2)}{SO(n-1) \times SU(2)_J}; \tag{6.41}$$

$$c\left(M_{\mathcal{N}=2, \mathfrak{J}_3^{2,n}}^5\right) = nc\left(M_{\mathcal{N}=2, \mathfrak{J}_3^{2,n}}^5\right) = \dim_{\mathbb{R}}\left(\widehat{M}_{\mathcal{N}=2, \mathfrak{J}_3^{2,n}}^5\right) = 3n + 3 = 9(q_{eff}(2, n) + 1) \tag{6.42}$$



where in (6.42) the definition (6.8) has been recalled.

### 6.1.1 $q = 0$ , $\mathfrak{J}_3^{2,2}$ (STU model)

Within the class  $\mathfrak{J}_3^{2,n}$  (6.3), the  $n = 2$  element

$$\mathfrak{J}_3^{2,2} \equiv \mathbb{R} \oplus \mathbf{\Gamma}_{1,1} \sim \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \quad (6.43)$$

deserves a more detailed analysis. After (6.5)-(6.7), its symmetry groups are

$$\mathfrak{L}(\mathfrak{J}_3^{2,2}) \equiv \mathfrak{qconf}(\mathfrak{J}_3^{2,2}) = \mathfrak{so}(4, 4); \quad (6.44)$$

$$\begin{aligned} \mathfrak{conf}(\mathfrak{J}_3^{2,2}) &= \mathfrak{aut}(\mathfrak{F}(\mathfrak{J}_3^{2,2})) = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(2, 2) \\ &\sim \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}); \end{aligned} \quad (6.45)$$

$$\mathfrak{str}_0(\mathfrak{J}_3^{2,2}) = \mathfrak{so}(1, 1) \oplus \mathfrak{so}(1, 1) = \mathfrak{so}(1, 1) \oplus \mathfrak{str}_0(\mathbf{\Gamma}_{1,1}), \quad (6.46)$$

Furthermore, the corresponding “effective” parameter  $q_{eff}$  (6.8) vanishes

$$\mathfrak{J}_3^{2,2} : q_{eff}(2, 2) = 0, \quad (6.47)$$

such that the “effective” dimension (6.9) of  $\mathfrak{J}_3^{2,2}$  takes value 3.

The rank-3 Euclidean *semi-simple* Jordan algebra  $\mathfrak{J}_3^{2,2}$  (6.43) corresponds to the so-called *STU* model [51, 52] of minimal supergravity (8 supersymmetries), whose *triality symmetry* is related to the complete factorization of  $\mathfrak{conf}(\mathfrak{J}_3^{2,2})$  (6.45). Furthermore, the vanishing value (6.47) of the “effective” parameter  $q_{eff}$  yields the identification of  $\mathfrak{J}_3^{2,2}$  as a Jordan algebra pertaining to the  $q = 0$  element of the  $q$ -parametrized “*exceptional sequence*” given by the second row of Table 1 (see *e.g.* [22]); indeed,  $\mathfrak{L}(\mathfrak{J}_3^{2,2}) = \mathfrak{so}(4, 4)$  (6.44) is a non-compact, real form (namely, the *split* form) of  $\mathfrak{so}(8)$ :

$$\mathfrak{L}^{q=0} \equiv \mathfrak{L}(\mathfrak{J}_3^{2,2}) = \mathfrak{qconf}(\mathfrak{J}_3^{2,2}) = \mathfrak{so}(4, 4). \quad (6.48)$$

It is here worth observing that  $\mathfrak{qconf}_c(\mathfrak{J}_3^{2,2}) = \mathfrak{L}_c^{q=0} = \mathfrak{so}(8)$  is the unique classical Lie algebra in the “*exceptional sequence*”, besides the limit case of  $\mathfrak{su}(3)$  (see also Footnote 2).

As pointed out above for the whole class  $\mathfrak{J}_3^{2,n}$ ,  $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{str}_0(\mathfrak{J}_3^{2,2}) = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{so}(1, 1) \oplus \mathfrak{so}(1, 1)$  is not maximally embedded into  $\mathfrak{L}(\mathfrak{J}_3^{2,2}) = \mathfrak{so}(4, 4)$ , but rather it can be embedded through a two-step chain of maximal symmetric embedding (*cfr.* (6.10)):

$$\begin{aligned} \mathfrak{so}(4, 4) &\supset \mathfrak{so}(3, 3)_0 \oplus \mathfrak{so}(1, 1)_0 \oplus \mathbf{6}_2 \oplus \mathbf{6}_{-2} \\ &\supset \mathfrak{sl}(3, \mathbb{R})_{0,0} \oplus \mathfrak{so}(1, 1)_{0,0} \oplus \mathfrak{so}(1, 1)_{0,0} \\ &\quad \oplus \mathbf{3}_{-4,0} \oplus \mathbf{3}'_{4,0} \oplus \mathbf{3}_{2,2} \oplus \mathbf{3}'_{-2,2} \oplus \mathbf{3}_{2,-2} \oplus \mathbf{3}'_{-2,-2}, \end{aligned} \quad (6.49)$$

or, at group level:

$$SO(4, 4) \supset SO(3, 3) \times SO(1, 1) \supset SL(3, \mathbb{R}) \times SO(1, 1) \times SO(1, 1); \quad (6.50)$$

$$\begin{aligned} \mathbf{28} &= \mathbf{15}_0 + \mathbf{1}_0 + \mathbf{6}_2 + \mathbf{6}_{-2} \\ &= \mathbf{8}_{0,0} + \mathbf{1}_{0,0} + \mathbf{3}_{-4,0} + \mathbf{3}'_{4,0} + \mathbf{1}_{0,0} + \mathbf{3}_{2,2} + \mathbf{3}'_{-2,2} + \mathbf{3}_{2,-2} + \mathbf{3}'_{-2,-2}. \end{aligned} \quad (6.51)$$

By observing that in

$$SL(3, \mathbb{R}) \times SO(1, 1) \quad : \quad \mathbf{6}_2 + \mathbf{6}_{-2} = (\mathbf{6}, \mathbf{2}); \quad (6.52)$$

$$SL(3, \mathbb{R}) \times SO(1, 1) \times SO(1, 1) \quad : \quad \mathbf{3}_{-4,0} + \mathbf{3}_{2,-2} + \mathbf{3}_{2,2} = (\mathbf{3}, \mathbf{2}_2 + \mathbf{1}_{-4}), \quad (6.53)$$

(6.49) and (6.51) can consistently be recast as the  $n = 2$  case of the general expressions (6.10) and (6.12) (analogous formulæ for the compact case hold). As mentioned, the “effective” dimension of  $\mathfrak{J}_3^{2,2}$  is  $n + 1 = 3$ , and the corresponding representation is *reducible* as  $\mathbf{2}_2 + \mathbf{1}_{-4}$  (6.53) with respect to  $Str_0(\mathfrak{J}_3^{2,2}) = SO(1, 1) \times SO(1, 1)$ .

Thus, the uplift of  $STU$  model to  $D = 5$ , based on  $\mathfrak{J}_3^{2,2}$ , is a *non-unified* theory [40]; in (6.53) (modulo redefinitions of the  $SO(1, 1)$  weights),  $\mathbf{2}_2$  corresponds to the graviphoton and the vector from the unique non-dilatonic vector multiplet (respectively with positive and negative signature in  $SO(1, 1)$ ), whereas  $\mathbf{1}_{-4}$  pertains to the vector from the dilatonic vector multiplet. We also note that, within the class  $\mathfrak{J}_3^{2,n}$ , only for  $\mathfrak{J}_3^{2,2}$  the total  $SO(1, 1)$ -weight of the 3-dimensional representation of the Jordan algebra  $\mathfrak{J}_3^{2,2}$  vanishes :  $2 \cdot 2 - 4 = 0$ .

Concerning the massless spectrum of the  $D = 5$  uplift of the  $STU$  model, the analysis goes as the case  $n = 2$  of the general treatment for  $\mathfrak{J}_3^{2,n}$  given in Subsec. 6.1; we briefly consider it below (as implied by the interpretation with 8 local supersymmetries, a  $D$ -independent hypermultiplet sector must be considered).

The *mcs* of  $\mathfrak{qconf}(\mathfrak{J}_3^{2,2}) = \mathfrak{so}(4, 4)$  and  $\mathfrak{str}_0(\mathfrak{J}_3^{2,2}) = \mathfrak{so}(1, 1) \oplus \mathfrak{so}(1, 1)$  respectively reads

$$mcs(\mathfrak{so}(4, 4)) = \mathfrak{so}(4) \oplus \mathfrak{so}(4) \sim \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)_H; \quad (6.54)$$

$$mcs(\mathfrak{so}(1, 1) \oplus \mathfrak{so}(1, 1)) = \emptyset. \quad (6.55)$$

(6.54) corresponds to the following maximal symmetric embeddings at group level:

$$SO(4, 4) \supset SO(4) \times SO(4) \sim SU(2) \times SU(2) \times SU(2) \times SU(2)_H; \quad (6.56)$$

$$\begin{aligned} \mathbf{28} &= (\mathbf{6}, \mathbf{1}) + (\mathbf{1}, \mathbf{6}) + (\mathbf{4}, \mathbf{4}) \\ &= (\mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3}) + (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}); \end{aligned} \quad (6.57)$$

However, the relevant maximal embedding must include the  $\mathfrak{su}(2)'$  algebra from the  $D$ -independent hypersector, as well:

$$\begin{aligned} \mathfrak{so}(4) \oplus \mathfrak{so}(4) \oplus \mathfrak{su}(2)' &\sim \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)_H \oplus \mathfrak{su}(2)' \\ &= \mathfrak{su}(2)_{\mathfrak{so}(4)} \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)_H \oplus \mathfrak{su}(2)' \\ &\quad \oplus \mathbf{3} \times \mathbf{1} \times \mathbf{1} \times \mathbf{1}; \end{aligned} \quad (6.58)$$

$$\begin{aligned} SO(4) \times SO(4) \times SU(2)' &\sim SU(2) \times SU(2) \times SU(2) \times SU(2)_H \times SU(2)' \\ &\supset SU(2)_{SO(4)} \times SU(2) \times SU(2)_H \times SU(2)'; \end{aligned} \quad (6.59)$$

$$\begin{aligned} (\mathbf{6}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{6}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}) &= (\mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1}) \\ &\quad + (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3}) \\ &= (\mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}) \\ &\quad + (\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1}), \end{aligned} \quad (6.60)$$

where  $SU(2)_{SO(4)}$  is diagonal into the first  $SU(2)$ 's on the r.h.s. of the isomorphism in the first line of (6.59):

$$SU(2)_{SO(4)} \sim SO(3) \subset_d SU(2) \times SU(2) \sim SO(4). \quad (6.61)$$

Consistently, the  $\mathcal{R}$ -symmetry of the  $D = 3$  dimensionally reduced  $STU$  model is enhanced from the quaternionic  $SU(2)_H$  (related to the  $c$ -map of the  $D = 4$  vector multiplets' scalar manifold) to  $SO(4) \sim SU(2)_H \times SU(2)'$ . On the other hand,  $SU(2)' \sim USp(2)$  is the  $\mathcal{R}$ -symmetry of the corresponding uplifted theory in  $D = 5$ .

Clearly, it holds that

$$\mathfrak{su}(2)' \cap \mathfrak{so}(4,4) = \emptyset \Rightarrow \mathfrak{su}(2)' \cap \mathfrak{su}(2)_J = \emptyset; \quad \mathfrak{su}(2)' \cap \mathfrak{sl}(3, \mathbb{R}) = \emptyset; \quad (6.62)$$

$$\mathfrak{su}(2)_{\mathfrak{so}(4)} \oplus \mathfrak{su}(2)_H \not\subseteq \mathfrak{sl}(3, \mathbb{R}), \quad (6.63)$$

where, as in general, the  $D = 5$  Ehlers Lie algebra  $\mathfrak{sl}(3, \mathbb{R})$  admits the massless spin algebra  $\mathfrak{su}(2)_J$  as maximal compact subalgebra.

From the embedding (6.54) and (6.56),  $(\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2})$  is the quadri-fundamental irrep.<sup>14</sup> of  $SO(4) \times SU(2) \times SU(2)_H \sim SU(2) \times SU(2) \times SU(2) \times SU(2)_H$ , in which the generators of the rank-4 symmetric quaternionic scalar manifold  $\frac{SO(4,4)}{SU(2) \times SU(2) \times SU(2) \times SU(2)_H}$  of the  $D = 3$  dimensionally reduced  $STU$  model sit. Trivially, the  $n = 2$  case of (6.21)-(6.22) yields that the generators of the rank-2 manifold  $SO(1,1) \times SO(1,1)$  of the  $D = 5$  uplift of  $STU$  model sit in the  $\mathbf{1} + \mathbf{1}$ . Under (6.58)-(6.59), it is then worth considering also the following branching:

$$\begin{aligned} SO(4) \times SO(4) \times SU(2)' &\sim SU(2) \times SU(2) \times SU(2) \times SU(2)_H \times SU(2)' \\ &\supset SU(2)_{SO(4)} \times SU(2) \times SU(2)_H \times SU(2)'; \end{aligned} \quad (6.64)$$

$$(\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1}) = (\mathbf{3}, \mathbf{2}, \mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}). \quad (6.65)$$

Some remarks are in order.

1. In order to identify the  $D = 5$  massless *spin group*  $SU(2)_J$ , a two-step procedure must be performed: **1.1]** one introduces the diagonal  $SU(2)_I$  into  $SU(2) \times SU(2)_H$  (*cfr.* (6.30)), such that (6.65) can be completed to the following chain:

$$\begin{aligned} SU(2) \times SU(2) \times SU(2) \times SU(2)_H \times SU(2)' &\supset SU(2)_{SO(4)} \times SU(2) \times SU(2)_H \times SU(2)' \\ &\supset SU(2)_{SO(4)} \times SU(2)_I \times SU(2)'; \end{aligned} \quad (6.66)$$

$$\begin{aligned} (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1}) &= (\mathbf{3}, \mathbf{2}, \mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}) \\ &= (\mathbf{3}, \mathbf{3}, \mathbf{1}) + (\mathbf{3}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}). \end{aligned} \quad (6.67)$$

**1.2]** Then, the  $D = 5$  massless *spin group*  $SU(2)_J$  is identified with the diagonal  $SU(2)_{II}$  into  $SU(2)_{SO(4)} \times SU(2)_I$  (*cfr.* (6.33)). Indeed, the chain (6.66)-(6.67) can be further completed as follows:

$$\begin{aligned} SU(2) \times SU(2) \times SU(2) \times SU(2)_H \times SU(2)' &\supset SU(2)_{SO(4)} \times SU(2) \times SU(2)_H \times SU(2)' \\ &\supset SU(2)_{SO(4)} \times SU(2)_I \times SU(2)' \\ &\supset SU(2)_J \times SU(2)'; \end{aligned} \quad (6.68)$$

$$\begin{aligned} (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1}) &= (\mathbf{3}, \mathbf{2}, \mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}) \\ &= (\mathbf{3}, \mathbf{3}, \mathbf{1}) + (\mathbf{3}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}) \\ &= (\mathbf{5}, \mathbf{1}) + (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}) + (\mathbf{3}, \mathbf{1}) + (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}). \end{aligned} \quad (6.69)$$

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<sup>14</sup>For application of  $(\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2})$  irrep. to the connection between QIT and supergravity, see *e.g.* [53, 54] (and Refs. therein).

The decomposition (6.69) corresponds to the massless bosonic spectrum of the  $D = 5$  uplift of the  $STU$  model (16 states): 1 graviton and 1 graviphoton (from the gravity multiplet), 1 dilatonic vector and 1 dilaton from the dilatonic vector multiplet, and 1 vector and 1 (real) scalar from the non-dilatonic vector multiplet. At the level of massless spectrum, the action of supersymmetry amounts to the following exchange of irreps.:

$$SO(n+2) \times SU(2) \times SU(2)_H \times SU(2)' : (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1})_B \longleftrightarrow (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{2})_F. \quad (6.70)$$

This can be realized by noticing that, under the chain (6.68) of maximal symmetric embeddings,  $(\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{2})$  decomposes as follows:

$$(\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{2}) = (\mathbf{3}, \mathbf{2}, \mathbf{1}, \mathbf{2}) + (\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2}) = (\mathbf{3}, \mathbf{2}, \mathbf{2}) + (\mathbf{1}, \mathbf{2}, \mathbf{2}) = (\mathbf{2}, \mathbf{2}) + (\mathbf{4}, \mathbf{2}) + (\mathbf{2}, \mathbf{2}), \quad (6.71)$$

thus reproducing the massless fermionic spectrum of the theory (16 states): 1  $SU(2)'$ -doublet of gravitinos, 1  $SU(2)'$ -doublet of dilatonic gauginos, and 1  $SU(2)'$ -doublet of gauginos from the non-dilatonic vector multiplet. Note that, consistently, bosons are  $\mathcal{R}$ -symmetry  $SU(2)'$ -singlets, whereas fermions fit into  $SU(2)'$ -doublets.

2.  $SU(2)_J$ , which commutes with  $SU(2)'$  inside  $SU(2) \times SU(2) \times SU(2) \times SU(2)_H \times SU(2)'$  (cfr. (6.68)), is the Kostant “principal”  $SU(2)$  (3.16) into the  $D = 5$  Ehlers group  $SL(3, \mathbb{R})$ :

$$SL(3, \mathbb{R}) \cap [SU(2)_{SO(4)} \times SU(2) \times SU(2)_H] = SU(2)_J. \quad (6.72)$$

3. As a consequence of the chain of maximal symmetric embeddings (6.50) and (6.68), the following (non-maximal, non-symmetric) manifold embedding holds:

$$\frac{SO(4, 4)}{SU(2) \times SU(2) \times SU(2) \times SU(2)_H} \supset SO(1, 1) \times SO(1, 1) \times \frac{SL(3, \mathbb{R})}{SU(2)_J}. \quad (6.73)$$

This has the trivial interpretation of embedding of the scalar manifold of the  $D = 5$  theory into the scalar manifold of the corresponding theory reduced to  $D = 3$ .

4. By setting  $n = 2$  in (6.40)-(6.42), one obtains:

$$M_{\mathcal{N}=2, \mathfrak{J}_3^{2,2}}^5 \equiv \frac{SO(4, 4)}{SO(1, 1) \times SO(1, 1) \times SL(3, \mathbb{R})_{\text{Ehlers}}}; \quad (6.74)$$

$$\widehat{M}_{\mathcal{N}=2, \mathfrak{J}_3^{2,2}}^5 \equiv \frac{SO(4) \times SO(4)}{SU(2)_J}; \quad (6.75)$$

$$c\left(M_{\mathcal{N}=2, \mathfrak{J}_3^{2,2}}^5\right) = nc\left(M_{\mathcal{N}=2, \mathfrak{J}_3^{2,2}}^5\right) = \dim_{\mathbb{R}}\left(\widehat{M}_{\mathcal{N}=2, \mathfrak{J}_3^{2,2}}^5\right) = 9, \quad (6.76)$$

consistent with the vanishing of  $q_{eff}$  for the STU model (cfr. (6.47)).

## 6.2 $\mathfrak{J}_3^{6,n} \equiv \mathbb{R} \oplus \mathbf{\Gamma}_{5,n-1}$

Let us now consider the class  $\mathfrak{J}_3^{6,n}$  (6.4) of *semi-simple* rank-3 Euclidean Jordan algebras. The relevant chain of embeddings reads:

$$\begin{aligned} \mathfrak{so}(8, n+2) &\supset \mathfrak{so}(3, 3) \oplus \mathfrak{so}(5, n-1) \oplus \mathbf{6} \times (\mathbf{n} + \mathbf{4}) \\ &\supset \mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{so}(5, n-1) \oplus \mathfrak{so}(1, 1) \\ &\quad \oplus \mathbf{3} \times ((\mathbf{n} + \mathbf{4})_2 + \mathbf{1}_{-4}) \oplus \mathbf{3}' \times ((\mathbf{n} + \mathbf{4})_{-2} + \mathbf{1}_4), \end{aligned} \quad (6.77)$$

or, at group level:

$$\begin{aligned} SO(8, n+2) &\supset SO(3, 3) \times SO(5, n-1) \\ &\supset SL(3, \mathbb{R}) \times SO(5, n-1) \times SO(1, 1); \end{aligned} \quad (6.78)$$

$$\begin{aligned} \mathbf{Adj}_{SO(8, n+2)} &= \mathbf{Adj}_{SO(3, 3)} + \mathbf{Adj}_{SO(5, n-1)} + (\mathbf{6}, \mathbf{n}+4) \\ &= \mathbf{Adj}_{SL(3, \mathbb{R})} + \mathbf{Adj}_{SO(1, 1)} + \mathbf{Adj}_{SO(5, n-1)} \\ &\quad + (\mathbf{3}, (\mathbf{n}+4)_2 + \mathbf{1}_{-4}) + (\mathbf{3}', (\mathbf{n}+4)_{-2} + \mathbf{1}_4), \end{aligned} \quad (6.79)$$

where the subscripts denote  $SO(1, 1)$ -weights.

According to (6.8) and (6.9), the “effective” dimension of  $\mathfrak{J}_3^{6, n}$  is  $n+5$ , and the corresponding Jordan algebra representation is *reducible* with respect to  $Str_0(\mathfrak{J}_3^{6, n}) = SO(1, 1) \times SO(5, n-1)$ , as given by (6.77) and (6.79):

$$\mathbf{n} + \mathbf{5} = (\mathbf{n} + \mathbf{4})_2 + \mathbf{1}_{-4}. \quad (6.80)$$

Thus, the  $\mathfrak{J}_3^{6, n}$ -related  $\mathcal{N} = 4$  (half-maximal),  $D = 5$  supergravity is *not unified*. In (6.80) (modulo redefinitions of the  $SO(1, 1)$  weights),  $(\mathbf{n} + \mathbf{4})_2$  corresponds to the 5 graviphotons and the  $n-1$  matter vectors (respectively with positive and negative signature in  $SO(5, n-1)$ ), whereas  $\mathbf{1}_{-4}$  pertains to the 2-form in the gravity multiplet.

The second line of (6.77) can be regarded as the extension of (2.5) to *semi-simple* rank-3 Euclidean Jordan algebras  $\mathfrak{J}_3^{6, n}$  (6.4). Its compact counterpart (which correspondingly generalizes (2.7)) reads

$$\begin{aligned} \mathfrak{so}(n+10) &\supset \mathfrak{so}(6) \oplus \mathfrak{so}(n+4) \oplus \mathbf{6} \times (\mathbf{n} + \mathbf{4}) \\ &\supset \mathfrak{su}(3) \oplus \mathfrak{so}(n+4) \oplus \mathfrak{u}(1) \\ &\quad \oplus \mathbf{3} \times ((\mathbf{n} + \mathbf{4})_2 + \mathbf{1}_{-4}) \oplus \overline{\mathbf{3}} \times ((\mathbf{n} + \mathbf{4})_{-2} + \mathbf{1}_4), \end{aligned} \quad (6.81)$$

with subscripts here denoting  $U(1)$ -charges.

At group level, the algebraic decompositions (6.77) and (6.81) respectively correspond to the second line of (6.78), which can be summarized by the non-maximal non-symmetric embedding

$$QConf(\mathfrak{J}_3^{6, n}) \supset SL(3, \mathbb{R}) \times Str_0(\mathfrak{J}_3^{6, n}), \quad (6.82)$$

and its compact counterpart:

$$SO(n+10) \supset SU(3) \times SO(n+4) \Leftrightarrow QConf_c(\mathfrak{J}_3^{6, n}) \supset SU(3) \times Str_{0, c}(\mathfrak{J}_3^{6, n}). \quad (6.83)$$

As mentioned,  $\mathfrak{J}_3^{6, n}$  pertains to *half-maximal* supergravity (16 supersymmetries): therefore, coupling is allowed to matter (vector) multiplets, with no  $D$ -independent hypermultiplet sector.

The *mcs* of  $\mathfrak{qconf}(\mathfrak{J}_3^{6, n}) = \mathfrak{so}(8, n+2)$  and  $\mathfrak{str}_0(\mathfrak{J}_3^{6, n}) = \mathfrak{so}(1, 1) \oplus \mathfrak{so}(5, n-1)$  respectively read

$$mcs(\mathfrak{so}(8, n+2)) = \mathfrak{so}(8) \oplus \mathfrak{so}(n+2); \quad (6.84)$$

$$mcs(\mathfrak{so}(1, 1) \oplus \mathfrak{so}(5, n-1)) = \mathfrak{so}(5) \oplus \mathfrak{so}(n-1) \sim \mathfrak{usp}(4) \oplus \mathfrak{so}(n-1), \quad (6.85)$$

corresponding to the following maximal symmetric embeddings at group level:

$$SO(8, n+2) \supset SO(8) \times SO(n+2); \quad (6.86)$$

$$\mathbf{Adj}_{SO(8, n+2)} = \mathbf{Adj}_{SO(n+2)} + \mathbf{Adj}_{SO(8)} + (\mathbf{8}_v, \mathbf{n} + \mathbf{2});$$

$$SO(1,1) \times SO(5, n-1) \supset SO(5) \times SO(n-1) \sim USp(4) \times SO(n-1); \quad (6.87)$$

$$\mathbf{1}_0 + \mathbf{Adj}_{SO(5, n-1), 0} = \mathbf{1} + \mathbf{Adj}_{SO(n-1)} + \mathbf{Adj}_{SO(5)} + (\mathbf{5}, \mathbf{n}-1), \quad (6.88)$$

where  $\mathbf{8}_v$  denotes the vector  $\mathbf{8}$  irrep. of  $SO(8)$ .

From the embedding (6.86),  $(\mathbf{8}_v, \mathbf{n} + \mathbf{2})$  is the bi-fundamental irrep. of  $SO(8) \times SO(n+2)$ , in which the generators of the symmetric scalar manifold  $\frac{SO(8, n+2)}{SO(8) \times SO(n+2)}$  of the  $D = 3$  half-maximal  $\mathfrak{J}_3^{6, n}$ -related supergravity sit. Furthermore, from the embedding (6.87), the generators of the symmetric scalar manifold  $SO(1,1) \times \frac{SO(5, n-1)}{USp(4) \times SO(n-1)}$  of the corresponding  $D = 5$  theory sit into the  $(\mathbf{1}, \mathbf{1}) + (\mathbf{5}, \mathbf{n}-1)$  of  $SO(5) \times SO(n-1) \sim USp(4) \times SO(n-1)$ . Thus, the relevant maximal embedding reads<sup>15</sup>

$$\begin{aligned} \mathfrak{so}(8) \oplus \mathfrak{so}(n+2) &= \mathfrak{so}(5) \oplus \mathfrak{so}(n-1) \oplus \mathfrak{su}(2)_I \oplus \mathfrak{su}(2)_{II} \\ &\oplus \mathbf{5} \times \mathbf{1} \times \mathbf{3} \times \mathbf{1} \oplus \mathbf{1} \times (\mathbf{n}-1) \times \mathbf{1} \times \mathbf{3}; \end{aligned} \quad (6.89)$$

$$SO(8) \times SO(n+2) \supset SO(5) \times SO(n-1) \times SU(2)_I \times SU(2)_{II}; \quad (6.90)$$

$$\begin{aligned} (\mathbf{28}, \mathbf{1}) + \mathbf{Adj}_{SO(n+2)} &= (\mathbf{10}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + \mathbf{Adj}_{SO(n-1)} \\ &+ (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3}) + (\mathbf{5}, \mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{n}-1, \mathbf{1}, \mathbf{3}); \end{aligned} \quad (6.91)$$

$$(\mathbf{8}_v, \mathbf{n} + \mathbf{2}) = (\mathbf{5}, \mathbf{n}-1, \mathbf{1}, \mathbf{1}) + (\mathbf{5}, \mathbf{1}, \mathbf{1}, \mathbf{3}) + (\mathbf{1}, \mathbf{n}-1, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{3}), \quad (6.92)$$

where

$$SO(8) \supset SO(5) \times SO(3) \sim USp(4) \times SU(2)_I; \quad (6.93)$$

$$\mathbf{8}_v = (\mathbf{5}, \mathbf{1}) + (\mathbf{1}, \mathbf{3});$$

$$\mathbf{28} = (\mathbf{10}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{5}, \mathbf{3});$$

$$SO(n+2) \supset SO(n-1) \times SO(3) \sim SO(n-1) \times SU(2)_{II}; \quad (6.94)$$

$$\mathbf{n} + \mathbf{2} = (\mathbf{n}-1, \mathbf{1}) + (\mathbf{1}, \mathbf{3});$$

$$\mathbf{Adj}_{SO(n+2)} = \mathbf{Adj}_{SO(n-1)} + (\mathbf{1}, \mathbf{3}) + (\mathbf{n}-1, \mathbf{3}).$$

Note that, differently from the analogous formula for *simple* rank-3 Euclidean Jordan algebras, the maximal embedding (6.89)-(6.90) is symmetric, as (6.93) and (6.94) are.

As consistently yielded by (6.86) and (6.87), the  $\mathcal{R}$ -symmetry of  $\mathcal{N} = 8$ ,  $D = 3$   $\mathfrak{J}_3^{6, n}$ -related half-maximal supergravity is  $SO(8)$ , whereas  $SO(5) \sim USp(4)$  is the  $\mathcal{R}$ -symmetry of the theory uplifted to  $D = 5$ .

1. Differently from the semi-simple class  $\mathfrak{J}_3^{2, n}$  treated above, the massless  $D = 5$  *spin group*  $SU(2)_J$  can be identified by a *one-step* procedure, with the diagonal  $SU(2)$  into  $SU(2)_I \times SU(2)_{II}$ :

$$SU(2)_J \subset_d SU(2)_I \times SU(2)_{II}. \quad (6.95)$$

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<sup>15</sup>We use a different convention on the branchings of the  $\mathbf{8}$ 's of  $SO(8)$  (with respect *e.g.* to [34]), namely:

$$SO(8) \supset SO(5) \times SO(3) \sim USp(4) \times SU(2)_I;$$

$$\mathbf{8}_v = (\mathbf{5}, \mathbf{1}) + (\mathbf{1}, \mathbf{3});$$

$$\mathbf{8}_s = (\mathbf{4}, \mathbf{2});$$

$$\mathbf{8}_c = (\mathbf{4}, \mathbf{2}).$$

This is one of the possible ones allowed by the  $SO(8)$  *triality*, and it can be regarded as the “physical” one, in which the vector  $\mathbf{8}_v$  of  $SO(8)$  decomposes into the fundamental (vector)  $\mathbf{5}$  of  $SO(5)$ .

Thus, (6.90) can be completed to the following chain:

$$\begin{aligned} SO(8) \times SO(n+2) &\supset SO(5) \times SO(n-1) \times SU(2)_I \times SU(2)_{II} \\ &\supset SO(5) \times SO(n-1) \times SU(2)_J; \end{aligned} \quad (6.96)$$

$$\begin{aligned} (\mathbf{8}_v, \mathbf{n}+2) &= (\mathbf{5}, \mathbf{n}-1, \mathbf{1}, \mathbf{1}) + (\mathbf{5}, \mathbf{1}, \mathbf{1}, \mathbf{3}) + (\mathbf{1}, \mathbf{n}-1, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{3}) \\ &= (\mathbf{5}, \mathbf{n}-1, \mathbf{1}) + (\mathbf{5}, \mathbf{1}, \mathbf{3}) + (\mathbf{1}, \mathbf{n}-1, \mathbf{3}) \\ &\quad + (\mathbf{1}, \mathbf{1}, \mathbf{5}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}). \end{aligned} \quad (6.97)$$

Indeed, the decomposition (6.97) corresponds to the massless bosonic spectrum of  $\mathcal{N} = 4$ ,  $D = 5$   $\mathfrak{J}_3^{4,n}$ -related half-maximal supergravity ( $8(n+2)$  states): 1 graviton, 1 2-form, 5 graviphotons and 1 real scalar from the gravity multiplet, and  $5(n-1)$  scalars and  $n-1$  vectors from the  $n-1$  matter (vector) multiplets. At the level of massless spectrum, the action of supersymmetry amounts to the following exchange of irreps.:

$$SO(8) \times SO(n+2) : (\mathbf{8}_v, \mathbf{n}+2) \xleftrightarrow{B} (\mathbf{8}_s, \mathbf{n}+2), \quad (6.98)$$

where  $\mathbf{8}_s$  is the chiral spinor<sup>16</sup> irrep. of  $SO(8)$ . This can be realized by noticing that, under the chain (6.97) of maximal symmetric embeddings,  $(\mathbf{8}_s, \mathbf{n}+2)$  decomposes as follows:

$$(\mathbf{8}_s, \mathbf{n}+2) = (\mathbf{4}, \mathbf{n}-1, \mathbf{2}, \mathbf{1}) + (\mathbf{4}, \mathbf{1}, \mathbf{2}, \mathbf{3}) = (\mathbf{4}, \mathbf{n}-1, \mathbf{2}) + (\mathbf{4}, \mathbf{1}, \mathbf{4}) + (\mathbf{4}, \mathbf{1}, \mathbf{2}), \quad (6.99)$$

thus reproducing the massless fermionic spectrum of the theory ( $8(n+2)$  states): 4 gravitinos and 4 spin 1/2 fermions (from the gravity multiplet), and  $4(n-1)$  gauginos from the  $n-1$  matter (vector) multiplets.

2.  $SU(2)_J$ , which commutes with  $USp(4) \times SO(n-1)$  inside  $SO(8) \times SO(n+2)$  (cfr. (6.96)), is the Kostant “principal”  $SU(2)$  (3.16) into the  $D = 5$  Ehlers group  $SL(3, \mathbb{R})$ :

$$SL(3, \mathbb{R}) \cap [SU(2)_I \times SU(2)_{II}] = SU(2)_J. \quad (6.100)$$

3. As a consequence of the chain of maximal symmetric embeddings (6.78) and (6.96), the following (non-maximal, non-symmetric) manifold embedding holds:

$$\frac{SO(8, n+2)}{SO(8) \times SO(n+2)} \supset SO(1, 1) \times \frac{SO(5, n-1)}{USp(4) \times SO(n-1)} \times \frac{SL(3, \mathbb{R})}{SU(2)_J}. \quad (6.101)$$

As usual, this has the trivial interpretation of embedding of the scalar manifold of the  $D = 5$  theory into the scalar manifold of the corresponding theory reduced to  $D = 3$ .

4. As resulting from the above treatment, and analogously to the “8 versus 24 supersymmetries” interpretation of  $\mathfrak{J}_3^{\mathbb{H}}$  discussed in Sec. 4, the main difference between the semi-simple classes  $\mathfrak{J}_3^{6,n}$  (6.4) and  $\mathfrak{J}_3^{2,n}$  (6.3) resides in the  $D$ -independent hypersector. In the former case, pertaining to *half-maximal* (16 supersymmetries) supergravity, such a sector is forbidden by supersymmetry; in the latter case, pertaining to *minimal* (8 supersymmetries) supergravity, *such a sector must be present for physical consistency*; as mentioned above, this hypersector is insensitive to dimensional reduction, and it is thus independent on the number  $D = 3, 4, 5, 6$  of space-time dimensions in which the theory with 8 supersymmetries is defined. The very same comments made at point 4 of Subsec. 3.2 also hold in this case, with (3.50) replaced by

$$SL(2, \mathbb{R}) \times \frac{SO(6, n)}{SO(6) \times SO(n)} \xrightarrow[D=4]{c_{hm}} \frac{SO(8, n+2)}{SO(8) \times SO(n+2)}, \quad (6.102)$$

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<sup>16</sup>Instead of  $\mathbf{8}_s$ , the conjugated chiral spinor  $\mathbf{8}_c$  can equivalently be chosen, as well.

where the coset on the l.h.s. is the vector multiplets' scalar manifold of the  $D = 4$  half-maximal theory; it is symmetric, as is its image  $\frac{SO(8,n+2)}{SO(8) \times SO(n+2)}$  through the “half-maximal” analogue  $c_{hm}$  of  $c$ -map.

5. In  $\mathcal{N} = 4$ ,  $D = 5$   $\mathfrak{J}_3^{6,n}$ -related supergravity, (3.25)-(3.27) respectively specify to<sup>17</sup> [10]

$$M_{\mathcal{N}=4, \mathfrak{J}_3^{6,n}}^5 \equiv \frac{SO(8, n+2)}{SO(1, 1) \times SO(5, n-1) \times SL(3, \mathbb{R})_{\text{Ehlers}}}; \quad (6.103)$$

$$\widehat{M}_{\mathcal{N}=4, \mathfrak{J}_3^{6,n}}^5 \equiv \frac{SO(8) \times SO(n+2)}{SO(5) \times SO(n-1) \times SU(2)_J}; \quad (6.104)$$

$$c\left(M_{\mathcal{N}=4, \mathfrak{J}_3^{6,n}}^5\right) = nc\left(M_{\mathcal{N}=4, \mathfrak{J}_3^{6,n}}^5\right) = \dim_{\mathbb{R}}\left(\widehat{M}_{\mathcal{N}=4, \mathfrak{J}_3^{6,n}}^5\right) = 3n + 15, \quad (6.105)$$

where in (6.105) the definition (6.8) has been recalled.

## 7 $\mathfrak{J}_3^{2,6} \sim \mathfrak{J}_3^{6,2}$ “Twin” Theories

As pointed out above, the presence or absence of a  $D$ -independent hypersector is implied by the physical (supergravity) interpretation of the model under consideration. In “(bosonic) twin” theories, sharing the very same bosonic sector, the  $D$ -independent hypersector can or cannot be considered, and in both cases the resulting supergravity theory (of course with different supersymmetry properties) is physically meaningful.

Besides the case of  $\mathfrak{J}_3^{\mathbb{H}}$ -related “(bosonic) twin” theories, treated in Sec. 4, another example is provided by the semi-simple rank-3 Euclidean Jordan algebra<sup>18</sup>

$$\mathfrak{J}_3^{2,6} \sim \mathfrak{J}_3^{6,2}, \quad (7.1)$$

given by the element  $n = 6$  of the class  $\mathfrak{J}_3^{2,n}$  (6.3):

$$8 \text{ susys} : \mathfrak{J}_3^{2,6} \equiv \mathbb{R} \oplus \mathbf{\Gamma}_{1,5}, \quad (7.2)$$

or by the element  $n = 6$  of the class  $\mathfrak{J}_3^{6,n}$  (6.4):

$$16 \text{ susys} : \mathfrak{J}_3^{6,2} \equiv \mathbb{R} \oplus \mathbf{\Gamma}_{5,1}. \quad (7.3)$$

The relevant symmetries read

$$\mathfrak{L}\left(\mathfrak{J}_3^{2,6}\right) \equiv \mathfrak{qconf}\left(\mathfrak{J}_3^{2,6}\right) = \mathfrak{so}(4, 8); \quad (7.4)$$

$$\mathfrak{conf}\left(\mathfrak{J}_3^{2,6}\right) = \mathfrak{aut}\left(\mathfrak{F}\left(\mathfrak{J}_3^{2,6}\right)\right) = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(2, 6); \quad (7.5)$$

$$\mathfrak{str}_0\left(\mathfrak{J}_3^{2,6}\right) = \mathfrak{so}(1, 1) \oplus \mathfrak{so}(1, 5) = \mathfrak{so}(1, 1) \oplus \mathfrak{str}_0\left(\mathbf{\Gamma}_{1,5}\right), \quad (7.6)$$

where  $\mathfrak{F}\left(\mathfrak{J}_3^{2,6}\right)$  denotes the *Freudenthal triple system* constructed over  $\mathfrak{J}_3^{2,6}$ . By recalling the definition (6.8), the corresponding “effective” parameter reads

$$q_{eff}(2, 6) = \frac{4}{3}, \quad (7.7)$$

such that the “effective” dimension of  $\mathfrak{J}_3^{2,6}$  is 7.

<sup>17</sup>With respect to the parameter  $m$  used in [10] (see *e.g.* Table 12 therein), we define  $n = m - 1$ .

<sup>18</sup>After the treatment of [46] (see also Refs. therein), the cases of  $\mathfrak{J}_3^{\mathbb{H}}$  and  $\mathfrak{J}_3^{2,6} \sim \mathfrak{J}_3^{6,2}$  are the unique cases of “(bosonic) twin” with symmetric scalar manifolds and with an interpretation in terms of rank-3 Euclidean Jordan algebras.



## 7.1 $\mathfrak{J}_3^{6,2}$ , 16 Supersymmetries

Let us start by considering  $\mathfrak{J}_3^{6,2}$ .

Consistently with Subsec. 6.2, it is associated to a theory with 16 supersymmetries, namely  $\mathcal{N} = 4$ ,  $D = 5$  half-maximal supergravity coupled to 1 matter (vector) multiplet, whose reduction to  $D = 3$  yields  $\mathcal{N} = 8$  supergravity coupled to 4 matter multiplets. No  $D$ -independent hypersector is allowed.

In this case, the relevant chain of embeddings reads  $(\mathfrak{so}(3, 3) \sim \mathfrak{sl}(4, \mathbb{R}); \mathfrak{so}(5, 1) \sim \mathfrak{su}^*(4))$ :

$$\begin{aligned} \mathfrak{so}(8, 4) &\supset \mathfrak{so}(3, 3) \oplus \mathfrak{so}(5, 1) \oplus \mathbf{6} \times \mathbf{6} \\ &\supset \mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{so}(5, 1) \oplus \mathfrak{so}(1, 1) \oplus \mathbf{3} \times (\mathbf{6}_2 + \mathbf{1}_{-4}) \oplus \mathbf{3}' \times (\mathbf{6}_{-2} + \mathbf{1}_4), \end{aligned} \quad (7.8)$$

or, at group level:

$$\begin{aligned} SO(8, 4) &\supset SO(3, 3) \times SO(5, 1) \\ &\supset SL(3, \mathbb{R}) \times SO(5, 1) \times SO(1, 1); \end{aligned} \quad (7.9)$$

$$\begin{aligned} \mathbf{66} &= (\mathbf{15}, \mathbf{1}) + (\mathbf{1}, \mathbf{15}) + (\mathbf{6}, \mathbf{6}) \\ &= (\mathbf{8}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{15})_0 + (\mathbf{3}, \mathbf{6}_2 + \mathbf{1}_{-4}) + (\mathbf{3}', \mathbf{6}_{-2} + \mathbf{1}_4), \end{aligned} \quad (7.10)$$

where the subscripts denote  $SO(1, 1)$ -weights.

The “effective” dimension 7 of  $\mathfrak{J}_3^{6,2}$  is *reducible* with respect to  $Str_0(\mathfrak{J}_3^{6,2}) = SO(1, 1) \times SO(5, 1) \sim SO(1, 1) \times SU^*(4)$ , as given by (7.8) and (7.10):

$$\mathbf{7} = \mathbf{6}_2 + \mathbf{1}_{-4}, \quad (7.11)$$

yielding that the  $\mathfrak{J}_3^{6,2}$ -related  $\mathcal{N} = 4$ ,  $D = 5$  theory is *non-unified*. In (7.11) (modulo redefinitions of the  $SO(1, 1)$  weights),  $\mathbf{6}_2$  corresponds to the 5 graviphotons and the unique matter vector (respectively with positive and negative signature in  $SO(5, 1)$ ), whereas  $\mathbf{1}_{-4}$  pertains to the 2-form in the gravity multiplet.

The *mcs* of  $\mathfrak{qconf}(\mathfrak{J}_3^{6,2}) = \mathfrak{so}(8, 4)$  and of  $\mathfrak{str}_0(\mathfrak{J}_3^{6,2}) = \mathfrak{so}(1, 1) \oplus \mathfrak{so}(5, 1)$  respectively read

$$mcs(\mathfrak{so}(8, 4)) = \mathfrak{so}(8) \oplus \mathfrak{so}(4) \sim \mathfrak{so}(8) \oplus \mathfrak{su}(2) \times \mathfrak{su}(2)_{(H)}; \quad (7.12)$$

$$mcs(\mathfrak{so}(1, 1) \oplus \mathfrak{so}(5, 1)) = \mathfrak{so}(5) \sim \mathfrak{usp}(4), \quad (7.13)$$

corresponding to the following maximal symmetric embeddings at group level:

$$SO(8, 4) \supset SO(8) \times SO(4) \sim SO(8) \times SU(2) \times SU(2)_{(H)}; \quad (7.14)$$

$$\begin{aligned} \mathbf{66} &= (\mathbf{28}, \mathbf{1}) + (\mathbf{1}, \mathbf{6}) + (\mathbf{8}_v, \mathbf{4}) \\ &= (\mathbf{28}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}) + (\mathbf{8}_v, \mathbf{2}, \mathbf{2}); \end{aligned} \quad (7.15)$$

$$SO(1, 1) \times SO(5, 1) \supset SO(5) \sim USp(4); \quad (7.16)$$

$$\mathbf{1}_0 + \mathbf{15}_0 = \mathbf{1} + \mathbf{10} + \mathbf{5}. \quad (7.17)$$

As for  $\mathfrak{J}_3^{\mathbb{H}}$  treated in Subsec. 4.1, the subscript “(H)” denotes the fact that  $SU(2)_{(H)}$  actually is the quaternionic  $SU(2)$  connection in the physical interpretation pertaining to 8 local supersymmetries (see below).

From the embedding (7.14),  $(\mathbf{8}_v, \mathbf{2}, \mathbf{2})$  is the tri-fundamental irrep. of  $SO(8) \times SU(2) \times SU(2)_{(H)}$ , in which the generators of the symmetric scalar manifold  $\frac{SO(8, 4)}{SO(8) \times SU(2) \times SU(2)_{(H)}}$  of  $\mathcal{N} = 8$ ,  $D = 3$   $\mathfrak{J}_3^{6,2}$ -related supergravity sit. On the other hand, (7.16) yields that  $\mathbf{1} + \mathbf{5}$  is the representation of  $SO(5)$  in

which the generators of the symmetric scalar manifold  $SO(1, 1) \times \frac{SO(5, 1)}{SO(5)}$  of the corresponding  $D = 5$  theory sit. Thus, the relevant maximal embedding reads

$$\begin{aligned} \mathfrak{so}(8) \oplus \mathfrak{so}(4) &\sim \mathfrak{so}(8) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)_{(H)} = \mathfrak{so}(5) \oplus \mathfrak{su}(2)_I \oplus \mathfrak{su}(2)_{II} \\ &\oplus \mathbf{5} \times \mathbf{3} \times \mathbf{1} \oplus \mathbf{1} \times \mathbf{1} \times \mathbf{3}; \end{aligned} \quad (7.18)$$

$$\begin{aligned} SO(8) \times SO(4) &\sim SO(8) \times SU(2) \times SU(2)_{(H)} \\ &\supset SO(5) \times SU(2)_I \times SU(2)_{II}; \end{aligned} \quad (7.19)$$

$$(\mathbf{28}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}) = (\mathbf{10}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}) + (\mathbf{5}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}); \quad (7.20)$$

$$(\mathbf{8}_v, \mathbf{2}, \mathbf{2}) = (\mathbf{5}, \mathbf{1}, \mathbf{1}) + (\mathbf{5}, \mathbf{1}, \mathbf{3}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{3}), \quad (7.21)$$

where (6.93) holds, and  $SU(2)_{II}$  is diagonal into  $SU(2) \times SU(2)_{(H)}$  (cfr. (6.61))

$$\begin{aligned} SO(4) &\sim SU(2) \times SU(2)_{(H)} \supset SO(3) \sim SU(2)_{II}; \\ (\mathbf{2}, \mathbf{2}) &= \mathbf{3} + \mathbf{1}; \quad (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) = \mathbf{3} + \mathbf{3}. \end{aligned} \quad (7.22)$$

Note that, differently from the analogous formula for *simple* rank-3 Euclidean Jordan algebras, the maximal embedding (7.18)-(7.19) is symmetric, as (6.93) and (7.22) are.

As consistently yielded by (7.14) and (7.16), the  $\mathcal{R}$ -symmetry of  $\mathcal{N} = 8$ ,  $D = 3$   $\mathfrak{J}_3^{6,2}$ -related supergravity is  $SO(8)$ , whereas  $SO(5) \sim USp(4)$  is the  $\mathcal{R}$ -symmetry of the same theory uplifted to  $D = (5)$ .

Some remarks are in order.

1. As pointed out in the analysis of  $\mathfrak{J}_3^{6,n}$  in Subsec. 6.2, the massless  $D = 5$  *spin group*  $SU(2)_J$  can be identified, by a *one-step* procedure, with the diagonal  $SU(2)$  into  $SU(2)_I \times SU(2)_{II}$ :

$$SU(2)_J \subset_d SU(2)_I \times SU(2)_{II}. \quad (7.23)$$

Thus, (7.19) can be completed to the following chain:

$$\begin{aligned} SO(8) \times SO(4) &\sim SO(8) \times SU(2) \times SU(2)_{(H)} \supset SO(5) \times SU(2)_I \times SU(2)_{II} \\ &\supset SO(5) \times SU(2)_J; \end{aligned} \quad (7.24)$$

$$\begin{aligned} (\mathbf{8}_v, \mathbf{4}) &= (\mathbf{8}_v, \mathbf{2}, \mathbf{2}) = (\mathbf{5}, \mathbf{1}, \mathbf{1}) + (\mathbf{5}, \mathbf{1}, \mathbf{3}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{3}) \\ &= (\mathbf{5}, \mathbf{1}) + (\mathbf{5}, \mathbf{3}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{1}, \mathbf{5}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{1}, \mathbf{1}). \end{aligned} \quad (7.25)$$

Indeed, the decomposition (7.25) corresponds to the massless bosonic spectrum of  $\mathcal{N} = 4$ ,  $D = 5$   $\mathfrak{J}_3^{4,2}$ -related half-maximal supergravity (32 states): 1 graviton, 1 2-form, 5 graviphotons and 1 real scalar from the gravity multiplet, and 5 real scalars and 1 vector from the unique matter (vector) multiplet. At the level of massless spectrum, the action of supersymmetry amounts to the following exchange of irreps.:

$$SO(8) \times SU(2) \times SU(2)_{(H)} : (\mathbf{8}_v, \mathbf{2}, \mathbf{2})_B \longleftrightarrow (\mathbf{8}_s, \mathbf{2}, \mathbf{2})_F, \quad (7.26)$$

where the the conjugated chiral spinor  $\mathbf{8}_c$  can be equivalently considered in place of  $\mathbf{8}_s$ , as well. This can be realized by noticing that, under the chain (7.25) of maximal symmetric embeddings,  $(\mathbf{8}_s, \mathbf{2}, \mathbf{2})$  decomposes as follows:

$$(\mathbf{8}_s, \mathbf{2}, \mathbf{2}) = (\mathbf{4}, \mathbf{2}, \mathbf{1}) + (\mathbf{4}, \mathbf{2}, \mathbf{3}) = (\mathbf{4}, \mathbf{2}) + (\mathbf{4}, \mathbf{4}) + (\mathbf{4}, \mathbf{2}), \quad (7.27)$$

thus reproducing the massless fermionic spectrum of the theory (32 states): 4 gravitinos and 4 spin 1/2 fermions (from gravity multiplet), and 4 gauginos from the unique matter (vector) multiplet.

2.  $SU(2)_J$ , which commutes with  $USp(4)$  inside  $SO(8) \times SU(2) \times SU(2)_{(H)}$  (cfr. (7.24)), is the Kostant “principal”  $SU(2)$  (3.16) into the  $D = 5$  Ehlers group  $SL(3, \mathbb{R})$ :

$$SL(3, \mathbb{R}) \cap [SU(2)_I \times SU(2)_{II}] = SU(2)_J. \quad (7.28)$$

3. As a consequence of the chain of maximal symmetric embeddings (7.9) and (7.24), the following (non-maximal, non-symmetric) manifold embedding holds:

$$\frac{SO(8, 4)}{SO(8) \times SU(2) \times SU(2)_{(H)}} \supset SO(1, 1) \times \frac{SO(5, 1)}{SO(5)} \times \frac{SL(3, \mathbb{R})}{SU(2)_J} \sim SO(1, 1) \times \frac{SU^*(4)}{USp(4)} \times \frac{SL(3, \mathbb{R})}{SU(2)_J}. \quad (7.29)$$

As usual, this has the trivial interpretation of embedding of the scalar manifold of the  $D = 5$  theory into the scalar manifold of the corresponding theory reduced to  $D = 3$ .

4. In  $\mathcal{N} = 4$ ,  $D = 5$   $\mathfrak{J}_3^{6,2}$ -related supergravity, (6.103)-(6.105) respectively specify to

$$M_{\mathcal{N}=4, \mathfrak{J}_3^{6,2}}^5 \equiv \frac{SO(8, 4)}{SO(1, 1) \times SO(5, 1) \times SL(3, \mathbb{R})_{\text{Ehlers}}}; \quad (7.30)$$

$$\widehat{M}_{\mathcal{N}=4, \mathfrak{J}_3^{6,2}}^5 \equiv \frac{SO(8) \times SO(4)}{SO(5) \times SU(2)_J}; \quad (7.31)$$

$$c\left(M_{\mathcal{N}=4, \mathfrak{J}_3^{6,2}}^5\right) = nc\left(M_{\mathcal{N}=4, \mathfrak{J}_3^{6,2}}^5\right) = \dim_{\mathbb{R}}\left(\widehat{M}_{\mathcal{N}=4, \mathfrak{J}_3^{6,2}}^5\right) = 21. \quad (7.32)$$

## 7.2 $\mathfrak{J}_3^{2,6}$ , 8 Supersymmetries

Let us now consider  $\mathfrak{J}_3^{2,6}$ .

Consistently with Subsec. 6.1, it is associated to a supergravity model with 8 supersymmetries, namely the  $\mathfrak{J}_3^{2,6}$ -based  $\mathcal{N} = 2$ ,  $D = 5$  supergravity coupled to 6 vector multiplets, and its dimensional reduction down to  $D = 3$ , which is coupled to 8 matter multiplets. For physical consistency, a  $D$ -independent hypermultiplet sector must be considered.

Clearly, the chains of embeddings (7.8), (7.9) and (7.10) also hold in this case, along with the considerations on the reducibility of the representation of  $\mathfrak{J}_3^{2,6}$  with respect to  $Str_0\left(\mathfrak{J}_3^{2,6}\right) = SO(1, 1) \times SO(1, 5)$  (cfr. (7.11)). Furthermore, the embeddings (7.12)-(7.17) also hold, with the brackets removed in the subscript “(H)”.

(7.11) is still true, but with a different interpretation, namely: the  $\mathfrak{J}_3^{2,6}$ -related  $\mathcal{N} = 2$ ,  $D = 5$  theory is *non-unified*, with  $\mathbf{6}_2$  corresponding to the graviphoton and the vectors from the 5 non-dilatonic vector multiplets (respectively with positive and negative signature in  $SO(1, 5)$ ; notice the consistent flip of signs with respect to the “twin”  $\mathfrak{J}_3^{6,2}$ -related theory), whereas  $\mathbf{1}_{-4}$  pertains to the vector from the dilatonic vector multiplet.

However, the relevant maximal embedding must include the  $\mathfrak{su}(2)'$  algebra from the  $D$ -independent

hypersector, as well:

$$\begin{aligned}
\mathfrak{so}(8) \oplus \mathfrak{so}(4) \oplus \mathfrak{su}(2)' &\sim \mathfrak{so}(8) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)_H \oplus \mathfrak{su}(2)' \\
&= \mathfrak{so}(5) \oplus \mathfrak{su}(2)_{\mathfrak{so}(8)} \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)_H \oplus \mathfrak{su}(2)' \\
&\oplus \mathbf{5} \times \mathbf{3} \times \mathbf{1} \times \mathbf{1} \times \mathbf{1};
\end{aligned} \tag{7.33}$$

$$\begin{aligned}
SO(8) \times SO(4) \times SU(2)' &\sim SO(8) \times SU(2) \times SU(2)_H \times SU(2)' \\
&\supset SO(5) \times SU(2)_{SO(8)} \times SU(2) \times SU(2)_H \times SU(2)'; \tag{7.34}
\end{aligned}$$

$$\begin{aligned}
(\mathbf{28}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{6}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}) &= (\mathbf{28}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3}) \\
&= (\mathbf{10}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1}) \\
&\quad + (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1}) \\
&\quad + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3}) + (\mathbf{5}, \mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1}).
\end{aligned} \tag{7.35}$$

As above, by  $SU(2)_{SO(8)}$  we denote the group commuting with  $SO(5)$  in the maximal symmetric embedding

$$SO(8) \supset SO(5) \times SO(3) \sim USp(4) \times SU(2)_{SO(8)}, \tag{7.36}$$

determining (7.33). Note that, differently from the analogous formula for *simple* rank-3 Euclidean Jordan algebras, the maximal embedding (7.33)-(7.34) is symmetric.

Note that the  $\mathcal{R}$ -symmetry of  $\mathcal{N} = 4$ ,  $D = 3$   $\mathfrak{J}_3^{2,6}$ -related supergravity is consistently enhanced from the quaternionic  $SU(2)_H$  (related to the  $c$ -map of the  $D = 4$  vector multiplets' scalar manifold) to  $SU(2)_H \times SU(2)'$ , as given by (3.31). On the other hand,  $SU(2)' \sim USp(2)$  is the  $\mathcal{R}$ -symmetry of the theory in  $D = 5$ .

Clearly, it holds that

$$\mathfrak{su}(2)' \cap \mathfrak{so}(4, 8) = \emptyset \Rightarrow \mathfrak{su}(2)' \cap \mathfrak{su}(2)_J = \emptyset; \mathfrak{su}(2)' \cap \mathfrak{sl}(3, \mathbb{R}) = \emptyset; \tag{7.37}$$

$$\mathfrak{su}(2)_{\mathfrak{so}(8)} \oplus \mathfrak{su}(2)_H \not\subseteq \mathfrak{sl}(3, \mathbb{R}), \tag{7.38}$$

where, as in general, the  $D = 5$  Ehlers Lie algebra  $\mathfrak{sl}(3, \mathbb{R})$  admits the massless spin algebra  $\mathfrak{su}(2)_J$  as maximal compact subalgebra.

From the embeddings considered above,  $(\mathbf{8}_v, \mathbf{2}, \mathbf{2})$  is the tri-fundamental irrep. of  $SO(8) \times SU(2) \times SU(2)_H$ , in which the generators of the rank-4 symmetric quaternionic scalar manifold  $\frac{SO(4,8)}{SO(8) \times SU(2) \times SU(2)_H}$  of  $\mathcal{N} = 4$ ,  $D = 3$   $\mathfrak{J}_3^{2,6}$ -related supergravity sit. Furthermore, from (7.17)  $\mathbf{1} + \mathbf{5}$  is the representation of  $SO(5)$  in which the generators of the rank-2 symmetric real special scalar manifold  $SO(1,1) \times \frac{SO(1,5)}{SO(5)}$  of the  $D = 5$  theory sit. Thus, under (7.33)-(7.34), it is worth considering also the following branching:

$$\begin{aligned}
SO(8) \times SO(4) \times SU(2)' &\sim SO(8) \times SU(2) \times SU(2)_H \times SU(2)' \\
&\supset SO(5) \times SU(2)_{SO(8)} \times SU(2) \times SU(2)_H \times SU(2)'; \tag{7.39}
\end{aligned}$$

$$(\mathbf{8}_v, \mathbf{2}, \mathbf{2}, \mathbf{1}) = (\mathbf{5}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{2}, \mathbf{2}, \mathbf{1}). \tag{7.40}$$

1. As for the class  $\mathfrak{J}_3^{2,n}$  (6.3) treated in Subsec. 6.1, and differently from the case of *simple* rank-3 Euclidean Jordan algebras, in order to identify the massless  $D = 5$  *spin group*  $SU(2)_J$  a *two-step* procedure must be performed: **1.1**] one introduces the diagonal  $SU(2)_I$  into  $SU(2) \times SU(2)_H$ ,

as given by (6.30), such that (7.40) can be completed to the following chain:

$$\begin{aligned} SO(8) \times SU(2) \times SU(2)_H \times SU(2)' &\supset SO(5) \times SU(2)_{SO(8)} \times SU(2) \times SU(2)_H \times SU(2)' \\ &\supset SO(5) \times SU(2)_{SO(8)} \times SU(2)_I \times SU(2)'; \end{aligned} \quad (7.41)$$

$$\begin{aligned} (\mathbf{8}_v, \mathbf{2}, \mathbf{2}, \mathbf{1}) &= (\mathbf{5}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{2}, \mathbf{2}, \mathbf{1}) \\ &= (\mathbf{5}, \mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{5}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}). \end{aligned} \quad (7.42)$$

**1.2]** Then, the massless  $D = 5$  spin group  $SU(2)_J$  can be identified with the diagonal  $SU(2)_{II}$  into  $SU(2)_{SO(8)} \times SU(2)_I$ :

$$SU(2)_J \equiv SU(2)_{II} \subset_d SU(2)_{SO(8)} \times SU(2)_I, \quad (7.43)$$

such that the chain (7.41)-(7.42) can be further completed as follows:

$$\begin{aligned} SO(8) \times SU(2) \times SU(2)_H \times SU(2)' &\supset SO(5) \times SU(2)_{SO(8)} \times SU(2) \times SU(2)_H \times SU(2)' \\ &\supset SO(5) \times SU(2)_{SO(8)} \times SU(2)_I \times SU(2)' \\ &\supset SO(5) \times SU(2)_J \times SU(2)'; \end{aligned} \quad (7.44)$$

$$\begin{aligned} (\mathbf{8}_v, \mathbf{2}, \mathbf{2}, \mathbf{1}) &= (\mathbf{5}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{2}, \mathbf{2}, \mathbf{1}) \\ &= (\mathbf{5}, \mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{5}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}) \\ &= (\mathbf{5}, \mathbf{3}, \mathbf{1}) + (\mathbf{5}, \mathbf{1}, \mathbf{1}) \\ &\quad + (\mathbf{1}, \mathbf{5}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}). \end{aligned} \quad (7.45)$$

The decomposition (7.45) corresponds to the massless bosonic spectrum of  $\mathcal{N} = 2$ ,  $D = 5$   $\mathfrak{J}_3^{2,6}$ -related supergravity : consistent with the fact that this theory is the “bosonic twin” of the  $\mathcal{N} = 4$ ,  $D = 5$   $\mathfrak{J}_3^{6,2}$ -related supergravity, they share the very same bosonic spectrum (32 states): 1 graviton and 1 graviphoton (belonging to the unique  $\mathcal{N} = 4$  vector multiplet) from the  $\mathcal{N} = 2$  gravity multiplet, 1 dilaton (corresponding to the scalar from the  $\mathcal{N} = 4$  gravity multiplet) and 1 dilatonic vector (which in the  $\mathcal{N} = 4$  interpretation corresponds to the 2-form in the gravity multiplet) from the  $\mathcal{N} = 2$  dilatonic vector multiplet, and 5 vectors (corresponding to the 5  $\mathcal{N} = 4$  graviphotons) and 5 real scalars (belonging to the  $\mathcal{N} = 4$  vector multiplet) from the 5 non-dilatonic  $\mathcal{N} = 2$  vector multiplets. Such states fit into

$$\mathcal{N} = 4 \text{ (16 susys)} : (\mathbf{8}_v, \mathbf{2}, \mathbf{2}) \text{ of } SO(8) \times SU(2) \times SU(2)_{(H)}; \quad (7.46)$$

$$\mathcal{N} = 2 \text{ (8 susys)} : (\mathbf{8}_v, \mathbf{2}, \mathbf{2}, \mathbf{1}) \text{ of } SO(8) \times SU(2) \times SU(2)_H \times SU(2)'. \quad (7.47)$$

On the other hand, the two theories consistently have different fermionic sectors; thus, the massless fermionic spectrum of  $\mathcal{N} = 2$ ,  $D = 5$   $\mathfrak{J}_3^{2,6}$ -related supergravity is not given by  $(\mathbf{8}_s, \mathbf{2}, \mathbf{2})$  (or  $(\mathbf{8}_c, \mathbf{2}, \mathbf{2})$ ), but rather by  $(\mathbf{8}_v, \mathbf{2}, \mathbf{1}, \mathbf{2})$ , of  $SO(8) \times SU(2) \times SU(2)_H \times SU(2)'$ . This can be realized by observing that, under (7.45), such an irrep. decomposes as:

$$\begin{aligned} (\mathbf{8}_v, \mathbf{2}, \mathbf{1}, \mathbf{2}) &= (\mathbf{5}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2}) + (\mathbf{1}, \mathbf{3}, \mathbf{2}, \mathbf{1}, \mathbf{2}) = (\mathbf{5}, \mathbf{1}, \mathbf{2}, \mathbf{2}) + (\mathbf{1}, \mathbf{3}, \mathbf{2}, \mathbf{2}) \\ &= (\mathbf{5}, \mathbf{2}, \mathbf{2}) + (\mathbf{1}, \mathbf{4}, \mathbf{2}) + (\mathbf{1}, \mathbf{2}, \mathbf{2}), \end{aligned} \quad (7.48)$$

thus corresponding to 5  $SU(2)'$ -doublets of non-dilatonic gauginos (from the 5 non-dilatonic vector multiplets), 1  $SU(2)'$ -doublet of gravitinos, and 1  $SU(2)'$ -doublet of dilatonic gauginos.

Thus, at the level of massless spectrum, in the minimal case the action of supersymmetry amounts to the following exchange of irreps.<sup>19</sup>:

$$SO(8) \times SU(2) \times SU(2)_H \times SU(2)' : (\mathbf{8}_v, \mathbf{2}, \mathbf{2}, \mathbf{1}) \longleftrightarrow (\mathbf{8}_v, \mathbf{2}, \mathbf{1}, \mathbf{2}), \quad (7.49)$$

to be contrasted with its analogue (7.26), holding in presence of 16 local supersymmetries. Note that, consistently, in the minimal interpretation bosons are  $\mathcal{R}$ -symmetry  $SU(2)'$ -singlets, whereas fermions fit into  $SU(2)'$ -doublets.

2.  $SU(2)_J$ , which commutes with  $SO(5) \times SU(2)'$  inside  $SO(8) \times SU(2) \times SU(2)_H \times SU(2)'$  (cfr. (7.44)), is the Kostant “principal”  $SU(2)$  (3.16) maximally embedded into the  $D = 5$  Ehlers group  $SL(3, \mathbb{R})$ :

$$SL(3, \mathbb{R}) \cap [SU(2)_{SO(8)} \times SU(2) \times SU(2)_H] = SU(2)_J. \quad (7.50)$$

3. As a consequence of the chain of maximal symmetric embeddings (7.9) and (7.44), the non-maximal, non-symmetric manifold embedding (7.29) holds, with a different interpretation in terms of 8 supersymmetries.
4. As resulting from the above treatment, and analogously to the “8 susys versus 24 susys” interpretation of  $\mathfrak{J}_3^{\mathbb{H}}$ , the main difference between  $\mathfrak{J}_3^{6,2}$  (16 supersymmetries) and  $\mathfrak{J}_3^{2,6}$  (8 supersymmetries) resides in the  $D$ -independent hypersector. In the former case, pertaining to half-maximal supergravity, such a sector is forbidden by supersymmetry. In the latter case, pertaining to minimal supergravity, such a sector must be present for physical consistency; as mentioned above, the hypersector is insensitive to dimensional reductions, and it is thus independent on the number  $D = 3, 4, 5, 6$  of space-time dimensions in which the theory with 8 supersymmetries is defined. The very same comments made at point 4 of Subsec. 3.2 also hold in this case, with (3.50) replaced by

$$SL(2, \mathbb{R}) \times \frac{SO(6, 2)}{SO(6) \times SO(2)} \xrightarrow{c} \frac{SO(8, 4)}{SO(8) \times \frac{SU(2) \times SU(2)_H}{D=3}}, \quad (7.51)$$

where the coset on the l.h.s. is the scalar manifold of the  $(\mathfrak{J}_3^{6,2} \sim \mathfrak{J}_3^{2,6})$ -related  $\mathcal{N} = 4$  (or  $\mathcal{N} = 2$ ),  $D = 4$  supergravity theory; it is symmetric, as is its image  $\frac{SO(8, 4)}{SO(8) \times SU(2) \times SU(2)_H}$  through  $c$ -map [43].

5. The very same formulæ (7.30)-(7.32) also hold in this case, but with the different interpretation (pertaining to 8 local supersymmetries) considered in this Subsection.

## 8 Conclusion

In the present investigation, we have spelled out the relation which exist between the Ehlers group in five dimensions and the rank-3, Euclidean (simple and semi-simple) Jordan algebra interpretation of supergravity theories, whose  $U$ -duality symmetry is given by the reduced structure group of the cubic norm the underlying Jordan algebra.

The massless spin (helicity) is enhanced to the Ehlers symmetry, and gets further enlarged to the so-called *super-Ehlers* symmetry [10] by the inclusion of the  $U$ -duality, which consistently encode the supermultiplet structure of the corresponding supergravity theory.

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<sup>19</sup>Consistently with the branching properties of  $SO(8)$  mentioned in Footnote 15, the irrep.  $(\mathbf{8}_s, \mathbf{2}, \mathbf{2}, \mathbf{1})$  (or  $(\mathbf{8}_c, \mathbf{2}, \mathbf{2}, \mathbf{1})$ ) of  $SO(8) \times SU(2) \times SU(2)_H \times SU(2)'$  does *not* occur as (massless) bosonic or fermionic representation pertaining to the  $D = 5$  theory.

It is interesting to note that the general  $D = 5$  *Jordan pair* non-symmetric embedding (1.4) is maximal for theories based on *simple* Jordan algebras, such as  $N = 16$  and 12 “pure” theories, as well as  $N = 4$  magical Maxwell-Einstein supergravities, whereas it is non-maximal for the  $N = 8$  and  $N = 4$  matter coupled theories based on *semi-simple* Jordan algebras. However, (1.4) always preserves the group rank, which is not related to supersymmetry but rather to the underlying Jordan algebra.

## Acknowledgements

We would like to thank Piero Truini and Shannon McCurdy for useful correspondence and discussions.

A.M. would like to thank the Department of Physics, University of California at Berkeley, where this project was completed, for kind hospitality and stimulating environment.

The work of S.F. has been supported by the ERC Advanced Grant no. 226455, Supersymmetry, Quantum Gravity and Gauge Fields (SUPERFIELDS).

The work of B. Z. has been supported in part by the Director, Office of Science, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231, and in part by NSF grant 30964-13067-44PHHXM.

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